Optimal Bandwidth Choice for Estimation of Inverse Conditional–Density–Weighted Expectations – Addendum –

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Abstract

This addendum provides the complete proof of Theorem (2.2) and its technical lemmas for the above paper.

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Appendix A: Main Proofs

Proof of Theorem 2.2

This is a long proof. It consists mostly of repetitive steps and calculations. Specifically, we look at the contribution to the MSE from each element on the right–hand side of (2.6). Firstly, let us denote

$$\begin{aligned}
\delta_1 &= E\left[\varpi_1\right], \\
\delta_2 &= E\left[\pi_2\left(\mathbf{U}\right) f_{\mathbf{U}}\left(\mathbf{U}\right)\right], \\
\delta_3 &= E\left[\pi_3\left(V, \mathbf{U}\right) f_{V\mathbf{U}}\left(V, \mathbf{U}\right)\right], \\
\delta_4 &= E\left[\pi_4\left(V, \mathbf{U}\right) f_{\mathbf{U}}\left(\mathbf{U}\right) f_{V\mathbf{U}}\left(V, \mathbf{U}\right)\right], \text{ and} \\
\delta_5 &= E\left[\pi_5\left(V, \mathbf{U}\right) f_{V\mathbf{U}}^2\left(V, \mathbf{U}\right)\right].
\end{aligned}$$

Then, by using the definitions in Section 2 and the properties of conditional expectations, it follows that

$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \eta.$$

In particular, we have for example,

$$\begin{aligned}
\delta_3 &= E\left[\pi_3\left(V, \mathbf{U}\right) f_{V\mathbf{U}}\left(V, \mathbf{U}\right)\right] \\
&= E\left[E\left[\left.\frac{\omega f_{\mathbf{U}}\left(\mathbf{U}\right)}{f_{V\mathbf{U}}^2\left(V, \mathbf{U}\right)}\right| V = v, \mathbf{U} = \mathbf{u}\right] f_{V\mathbf{U}}\left(V, \mathbf{U}\right)\right] \\
&= E\left[E\left[\left.\frac{\omega f_{\mathbf{U}}\left(\mathbf{U}\right)}{f_{V\mathbf{U}}\left(V, \mathbf{U}\right)}\right| V = v, \mathbf{U} = \mathbf{u}\right]\right] \\
&= E\left[\frac{\omega f_{\mathbf{U}}\left(\mathbf{u}\right)}{f_{V\mathbf{U}}\left(v, \mathbf{u}\right)}\right] = E\left[\frac{\omega}{f_{V|\mathbf{U}}\left(v|\,\mathbf{u}\right)}\right] \\
&= \eta.
\end{aligned}$$

Similar results hold for the rest.

Therefore, we can write $E[\|\widehat{\eta}(h)-\eta\|^2]$ as,

$$E\left[\|\widehat{\eta}(h)-\eta\|^2\right] = E\left[\left\|\widehat{\delta}_1-\delta_1\right\|^2\right] \tag{A-1}$$

$$+4E\left[\left\|\widehat{\delta}_2(h)-\delta_2\right\|^2\right] \tag{A-2}$$

$$+4E\left[\left\|\widehat{\delta}_3(h)-\delta_3\right\|^2\right] \tag{A-3}$$

$$+E\left[\left\|\widehat{\delta}_4(h)\right\|^2\right] \tag{A-4}$$

$$+E\left[\left\|\widehat{\delta}_5(h)\right\|^2\right] \tag{A-5}$$

$$+4E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_2(h)-\delta_2\right\rangle\right] \tag{A-6}$$

$$-4E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_3(h)-\delta_3\right\rangle\right] \tag{A-7}$$

$$-2E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_4(h)-\delta_4\right\rangle\right] \tag{A-8}$$

$$+2E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_5(h)-\delta_5\right\rangle\right] \tag{A-9}$$

$$-8E\left[\left\langle\widehat{\delta}_2(h)-\delta_2,\widehat{\delta}_3(h)-\delta_3\right\rangle\right] \tag{A-10}$$

$$-4E\left[\left\langle\widehat{\delta}_2(h)-\delta_2,\widehat{\delta}_4(h)-\delta_4\right\rangle\right] \tag{A-11}$$

$$+4E\left[\left\langle\widehat{\delta}_2(h)-\delta_2,\widehat{\delta}_5(h)-\delta_5\right\rangle\right] \tag{A-12}$$

$$+4E\left[\left\langle\widehat{\delta}_3(h)-\delta_3,\widehat{\delta}_4(h)-\delta_4\right\rangle\right] \tag{A-13}$$

$$-4E\left[\left\langle\widehat{\delta}_3(h)-\delta_3,\widehat{\delta}_5(h)-\delta_5\right\rangle\right] \tag{A-14}$$

$$-2E\left[\left\langle\widehat{\delta}_4(h)-\delta_4,\delta_5(h)-\delta_5\right\rangle\right]. \tag{A-15}$$

In what follows, we look at each of the 15 terms above.

Term: $E\left[\left\|\widehat{\delta}_1-\delta_1\right\|^2\right]$

Firstly, notice that $E[\|\ \widehat{\delta}_1-\delta_1\ \|^2] = E[\|\ N^{-1}\sum_{i=1}^N \varepsilon_{1i}\ \|^2] + E[\|\ N^{-1}\sum_{i=1}^N \widetilde{\pi}_{1i}-\delta_1\ \|^2]$, and clearly

$$E\left[\varepsilon_{1i}^\top\varepsilon_{1j}\middle|(V_1,\mathbf{U}_1),\ldots,(V_N,\mathbf{U}_N)\right] = \begin{cases}\sigma_1^2(V_i,\mathbf{U}_i), & i=j,\\ 0, & i\neq j.\end{cases}$$

so

$$E\left[\left\|\widehat{\delta}_1-\delta_1\right\|^2\right] = O\left(\frac{1}{N}\right). \tag{A-16}$$

Terms: $E\left[\left\|\widehat{\delta}_2(h)-\delta_2\right\|^2\right]$ **and** $E\left[\left\|\widehat{\delta}_3(h)-\delta_3\right\|^2\right]$

We have

$$E\left[\left\|\widehat{\delta}_2(h)-\delta_2\right\|^2\right]=E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{2i}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)\right\|^2\right]+E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}(\zeta_{2i}-E[\zeta_{2i}])\right\|^2\right]$$
$$+\left\|\frac{1}{N}\sum_{i=1}^{N}E[\zeta_{2i}]-\delta_2\right\|^2,\text{ and}$$
$$E\left[\left\|\widehat{\delta}_3(h)-\delta_3\right\|^2\right]=E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{3i}\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\|^2\right]+E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}(\zeta_{3i}-E[\zeta_{3i}])\right\|^2\right]$$
$$+\left\|\frac{1}{N}\sum_{i=1}^{N}E[\zeta_{3i}]-\delta_3\right\|^2,$$

with $\zeta_{2i}\equiv\pi_2(\mathbf{u}_i)\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)$, and $\zeta_{3i}\equiv\pi_3(v_i,\mathbf{u}_i)\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)$. They are such that $E[\zeta_{2i}]=E[\zeta_{21}]=q_2$, and $E[\zeta_{3i}]=E[\zeta_{31}]=q_3$, where we use the definitions $q_2\equiv E[\pi_2(\mathbf{U}_1)\times\mathcal{K}_h(\mathbf{U}_2-\mathbf{U}_1)]$, $q_3\equiv E[\pi_3(V_1,\mathbf{U}_1)W_h(V_2-V_1)\mathcal{K}_h(\mathbf{U}_2-\mathbf{U}_1)]$, and notation $W_h(c)=h^{-1}W(c/h)$ and $\mathcal{K}_h(\mathbf{c})=h^{-(d-1)}\mathcal{K}(h^{-1}\mathbf{c})$.

Notice that,

$$E\left[\varepsilon_{2i}^{\top}\varepsilon_{2j}\middle|\mathbf{U}_1,\ldots,\mathbf{U}_N\right]=\begin{cases}\sigma_2^2(\mathbf{U}_i), & i=j,\\ 0, & i\neq j.\end{cases},$$
$$E\left[\varepsilon_{3i}^{\top}\varepsilon_{3j}\middle|(V_1,\mathbf{U}_1),\ldots,(V_N,\mathbf{U}_N)\right]=\begin{cases}\sigma_3^2(V_i,\mathbf{U}_i), & i=j,\\ 0, & i\neq j.\end{cases},$$

and we write

$$E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{3i}\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\|^2\right]=\frac{1}{N^2}\sum_{i=1}^{N}E\left[\sigma_3^2(v_i,\mathbf{u}_i)\left\|\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\|^2\right]$$
$$=\frac{1}{N(N-1)^2}\int\sigma_3^2(v,\mathbf{u})E\left[\left\|\sum_{t=2}^{N}W_h(V_t-v)\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\right\|^2\right]f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}.$$

We have $E[\|\sum_{t=2}^{N}W_h(V_t-v)\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\|^2]=(N-1)E(\|W_h(V_1-v)\mathcal{K}_h(\mathbf{U}_1-\mathbf{u})\|^2)+(N-1)(N-2)\|E(W_h(V_1-v)\mathcal{K}_h(\mathbf{U}_1-\mathbf{u}))\|^2$. If follows from Lemmas B-1 and B-2 that

$$E\left[\left\|\sum_{t=2}^{N}W_h(V_t-v)\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\right\|^2\right]=(N-1)\left[C_{W\mathcal{K}}f_{V\mathbf{U}}(v,\mathbf{u})h^{-d}+\psi_{W\mathcal{K}}(h,(v,\mathbf{u}))\right]$$
$$+(N-1)(N-2)\left\|f_{V\mathbf{U}}(v,\mathbf{u})+h^{P}S_{W\mathcal{K}}(v,\mathbf{u})+\beta_{W\mathcal{K}}(h,(v,\mathbf{u}))\right\|^2.$$

Thence

$$E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{3i}\widehat{f}_{V\mathbf{U}}\left(v_i,\mathbf{u}_i\right)\right\|^2\right] = \frac{1}{N}\int\sigma_3^2\left(v,\mathbf{u}\right)f_{V\mathbf{U}}^3\left(v,\mathbf{u}\right)dvd\mathbf{u}$$
$$+\frac{C_{W\mathcal{K}}}{N^2h^d}\int\sigma_3^2\left(v,\mathbf{u}\right)f_{V\mathbf{U}}^2\left(v,\mathbf{u}\right)dvd\mathbf{u}$$
$$+O\left(\frac{h^P}{N}+\frac{1}{N^2}\right)+o\left(\frac{1}{N^2h^d}\right),\ N\to\infty, \tag{A-17}$$

by similar arguments, we also show that $E[\| N^{-1}\sum_{i=1}^{N}\varepsilon_{2i}\widehat{f}_{\mathbf{U}}\left(\mathbf{u}_i\right)\|^2] = N^{-1}\int\sigma_2^2\left(\mathbf{u}\right)\times f_{\mathbf{U}}^3\left(\mathbf{u}\right)d\mathbf{u}+ N^{-2}h^{-(d-1)}C_K\int\sigma_2^2\left(\mathbf{u}\right)f_{\mathbf{U}}^2\left(\mathbf{u}\right)d\mathbf{u}+O\left(N^{-1}h^P+N^{-2}\right)+o(N^{-2}h^{-(d-1)}),\ N\to\infty$.

Next,

$$E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_{li}-E\left[\zeta_{li}\right]\right)\right\|^2\right] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left[\langle\zeta_{li},\zeta_{lj}\rangle\right]-\left\|E\left[\zeta_{l1}\right]\right\|^2$$
$$=\frac{1}{N^2}\sum_{i=1}^{N}E\left[\left\|\zeta_{li}\right\|^2\right]+\frac{1}{N^2}\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}E\left[\langle\zeta_{li},\zeta_{lj}\rangle\right]-\left\|E\left[\zeta_{l1}\right]\right\|^2$$
$$=\frac{1}{N}E\left[\left\|\zeta_{l1}\right\|^2\right]+\frac{N-1}{N}E\left[\langle\zeta_{l1},\zeta_{l2}\rangle\right]-\left\|q_l\right\|^2, \tag{A-18}$$

for $l=2,3$.

Now, we have

$$E\left[\left\|\zeta_{31}\right\|^2\right] = \frac{1}{\left(N-1\right)^2}E\left[\left\|\pi_3\left(V_1,\mathbf{U}_1\right)\right\|^2\sum_{t=2}^{N}\sum_{s=2}^{N}W_h\left(V_t-V_1\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{U}_1\right)\times\right.$$
$$\left.W_h\left(V_s-V_1\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{U}_1\right)\right]$$
$$=\frac{1}{\left(N-1\right)^2}\int\left\|\pi_3\left(v,\mathbf{u}\right)\right\|^2E\left[\sum_{t=2}^{N}\sum_{s=2}^{N}W_h\left(V_t-v\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)\times\right.$$
$$\left.W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right]f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u},$$

where

$$E\left[\sum_{t=2}^{N}\sum_{s=2}^{N}W_h\left(V_t-v\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right]$$
$$=\sum_{t=2}^{N}E\left[\left\|W_h\left(V_t-v\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)\right\|^2\right]$$
$$+\sum_{\substack{t=2\\t\neq s}}^{N}\sum_{s=2}^{N}E\left[W_h\left(V_t-v\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)\right]E\left[W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right]$$
$$=(N-1)\,E\left[\left\|W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right\|^2\right]+(N-1)(N-2)\left\|E\left[W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]\right\|^2.$$

It follows from Lemmas B-1 and B-2 that

$$E\left[\sum_{t=2}^{N}\sum_{s=2}^{N}W_h\left(V_t-v\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right]$$
$$=(N-1)\left(C_{W\mathcal{K}}f_{V\mathbf{U}}\left(v,\mathbf{u}\right)h^{-d}+\psi_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right)$$
$$+(N-1)(N-2)\left\|f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^PS_{W\mathcal{K}}\left(v,\mathbf{u}\right)+\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right\|^2.$$

Hence

$$\frac{1}{N}E\left[\left\|\zeta_{31}\right\|^2\right]=\frac{1}{N}\int\left\|\pi_3\left(v,\mathbf{u}\right)\right\|^2f_{v\mathbf{u}}^3\left(v,\mathbf{u}\right)dv\,d\mathbf{u}+\frac{C_{W\mathcal{K}}}{N^2h^d}\int\left\|\pi_3\left(v,\mathbf{u}\right)\right\|^2f_{v\mathbf{u}}^2\left(v,\mathbf{u}\right)dv\,d\mathbf{u}$$
$$+O\left(N^{-1}h^P+N^{-2}\right)+o\left(N^{-2}h^{-d}\right),\text{ as }N\to\infty,\tag{A-19}$$

by Assumption (A3), and the properties of $\psi_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)$ and $\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)$. Similarly, we show that

$$\frac{1}{N}E\left[\left\|\zeta_{21}\right\|^2\right]=\frac{1}{N}\int\left\|\pi_2\left(\mathbf{u}\right)\right\|^2f_{\mathbf{u}}^3\left(\mathbf{u}\right)d\mathbf{u}+\frac{C_{\mathcal{K}}}{N^2h^{d-1}}\int\left\|\pi_2\left(\mathbf{u}\right)\right\|^2f_{\mathbf{u}}^2\left(\mathbf{u}\right)d\mathbf{u}$$
$$+O\left(N^{-1}h^P+N^{-2}\right)+o\left(N^{-2}h^{-(d-1)}\right),\text{ as }N\to\infty.\tag{A-20}$$

Now consider the term

$$\frac{N-1}{N}E\left[\left\langle\zeta_{21},\zeta_{22}\right\rangle\right]=\frac{1}{N(N-1)}E\left[\left\langle\pi_2\left(\mathbf{U}_1\right),\pi_2\left(\mathbf{U}_2\right)\right\rangle\right.$$
$$\left.\times\sum_{\substack{t=1\\t\neq1,s\neq2}}^{N}\sum_{s=1}^{N}K_h\left(\mathbf{U}_t-\mathbf{U}_1\right)K_h\left(\mathbf{U}_s-\mathbf{U}_2\right)\right]$$
$$=\frac{1}{N(N-1)}\sum_{\substack{t=1\\t\neq1,s\neq2}}^{N}\sum_{s=1}^{N}\Delta_{2,ts},$$

where $\Delta_{2,ts}=E\left[\left\langle\pi_2\left(\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{U}_1\right),\pi_2\left(\mathbf{U}_2\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{U}_2\right)\right\rangle\right]$. Similarly, $N^{-1}(N-1)\times E\left[\left\langle\zeta_{31},\zeta_{32}\right\rangle\right]=N^{-1}(N-1)^{-1}\sum_{\substack{t=1\\t\neq1,s\neq2}}^{N}\sum_{s=1}^{N}\Delta_{3,ts}$, with

$$\Delta_{3,ts} = E[\langle \pi_3 (V_1, \mathbf{U}_1) W_h (V_t - V_1) \mathcal{K}_h (\mathbf{U}_t - \mathbf{U}_1), \pi_3 (V_2, \mathbf{U}_2) W_h(V_s - V_2)\mathcal{K}_h (\mathbf{U}_s - \mathbf{U}_2)\rangle].$$

Furthermore, for $l = 2, 3$, we write

$$\Delta_{l,ts} = \begin{cases} \mathcal{B}_{l,I} & ; \quad s = t, \\ \mathcal{B}_{l,II} & ; \quad s \neq t,\ t \neq 2,\ s \neq 1, \\ \mathcal{B}_{l,III} & ; \quad s = t \ \&\ t = 2,\ s \neq 1 \text{ or } t \neq 2,\ s = 1, \\ \mathcal{B}_{l,IV} & ; \quad s \neq t,\ t = 2,\ s = 1. \end{cases}$$

Here we make the following definitions:

$$\begin{aligned} \mathcal{B}_{2,I} &= E\left[\left\| E\left[\pi_2 (\mathbf{U}_1) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_1) \middle| \mathbf{U}_3\right]\right\|^2\right] &&; \\ \mathcal{B}_{2,II} &= \left\| E\left[\pi_2 (\mathbf{U}_1) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_1)\right]\right\|^2 \equiv \|q_2\|^2 &&; \\ \mathcal{B}_{2,III} &= E\left[\langle \pi_2 (\mathbf{U}_1), \pi_2 (\mathbf{U}_2)\rangle \mathcal{K}_h (\mathbf{U}_2 - \mathbf{U}_1) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_2)\right] &&; \\ \mathcal{B}_{2,IV} &= E\left[\langle \pi_2 (\mathbf{U}_1), \pi_2 (\mathbf{U}_2)\rangle \left\|\mathcal{K}_h (\mathbf{U}_1 - \mathbf{U}_2)\right\|^2\right] &&; \end{aligned}$$

$$\begin{aligned} \mathcal{B}_{3,I} &= E\left[\left\| E\left[\pi_3 (V_1, \mathbf{U}_1) W_h (V_3 - V_1) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_1) \middle| V_3, \mathbf{U}_3\right]\right\|^2\right] &&; \\ \mathcal{B}_{3,II} &= \left\| E\left[\pi_3 (V_1, \mathbf{U}_1) W_h (V_3 - V_1) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_1)\right]\right\|^2 \equiv \|q_3\|^2 &&; \\ \mathcal{B}_{3,III} &= E\left[\langle \pi_3 (V_1, \mathbf{U}_1), \pi_3 (V_2, \mathbf{U}_2)\rangle W_h (V_2 - V_1) \mathcal{K}_h (\mathbf{U}_2 - \mathbf{U}_1) \times \right. \\ &\qquad \left. W_h (V_3 - V_2) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_2)\right] &&; \\ \mathcal{B}_{3,IV} &= E\left[\langle \pi_3 (V_1, \mathbf{U}_1), \pi_3 (V_2, \mathbf{U}_2)\rangle \left\|W_h (V_1 - V_2) \mathcal{K}_h (\mathbf{U}_1 - \mathbf{U}_2)\right\|^2\right], \text{ and} \end{aligned}$$

Therefore, we are able to write

$$\begin{aligned} \frac{N-1}{N} E\left[\langle \zeta_{21}, \zeta_{22}\rangle\right] = \frac{1}{N(N-1)} \Big\{ & (N-2) \mathcal{B}_{2,I} + \left(N^2 - 5N + 6\right) \|q_2\|^2 \\ & + 2(N-2) \mathcal{B}_{2,III} + \mathcal{B}_{2,IV} \Big\}, \end{aligned} \tag{A-21}$$

$$\begin{aligned} \frac{N-1}{N} E\left[\langle \zeta_{31}, \zeta_{32}\rangle\right] = \frac{1}{N(N-1)} \Big\{ & (N-2) \mathcal{B}_{3,I} + \left(N^2 - 5N + 6\right) \|q_3\|^2 \\ & + 2(N-2) \mathcal{B}_{3,III} + \mathcal{B}_{3,IV} \Big\}. \end{aligned} \tag{A-22}$$

We now show the working of (A-22), because (A-21)'s is the same. As $N \to \infty$, it follows that

$$\begin{aligned} \frac{N-1}{N} E\left[\langle \zeta_{31}, \zeta_{32}\rangle\right] &= \frac{1}{N}\left[\mathcal{B}_{3,I} + 2\mathcal{B}_{3,III}\right] + \|q_3\|^2 \left[1 - \frac{5}{N}\right] + \frac{1}{N^2} \mathcal{B}_{3,IV} \\ &+ \frac{1}{N^2}\left[6\|q_3\|^2 - 2\mathcal{B}_{3,I} - 4\mathcal{B}_{3,III}\right] + o\left(N^{-2}\right) \\ &= \frac{3}{N} \int \|\pi_3 (v, \mathbf{u})\|^2 f_{\mathbf{U}}^3 (v, \mathbf{u})\, dv d\mathbf{u} + \|q_3\|^2 \left[1 - \frac{5}{N}\right] \\ &+ \frac{C_{W\mathcal{K}}}{N^2 h^d} \int \|\pi_3 (v, \mathbf{u})\|^2 f_{V\mathbf{U}}^2 (v, \mathbf{u})\, dv d\mathbf{u} \\ &+ O\left(N^{-2}\right) + o\left(N^{-2} h^{-d} + N^{-1} h^{P}\right), \end{aligned} \tag{A-23}$$

where the last equality follows from Lemmas B-1 and B-6. We now put together (A-19), (A-23) in (A-18) and obtain,

$$E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_{3i}-E\left[\zeta_{3i}\right]\right)\right\|^{2}\right]=\frac{1}{N}\left[4\int\left\|\pi_{3}\left(v,\mathbf{u}\right)\right\|^{2}f_{V\mathbf{U}}^{3}\left(v,\mathbf{u}\right)dvd\mathbf{u}-5\left\|\delta_{3}\right\|^{2}\right]$$
$$+\frac{2C_{W\mathcal{K}}}{N^{2}h^{d}}\int\left\|\pi_{3}\left(v,\mathbf{u}\right)\right\|^{2}f_{V\mathbf{U}}^{2}\left(v,\mathbf{u}\right)dvd\mathbf{u}$$
$$+O\left(N^{-2}\right)+o\left(N^{-2}h^{-d}+N^{-1}h^{P}\right),\text{ as }N\rightarrow\infty,\qquad\text{(A-24)}$$

similarly

$$E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_{2i}-E\left[\zeta_{2i}\right]\right)\right\|^{2}\right]=\frac{1}{N}\left[4\int\left\|\pi_{2}\left(\mathbf{u}\right)\right\|^{2}f_{\mathbf{U}}^{3}\left(\mathbf{u}\right)d\mathbf{u}-5\left\|\delta_{2}\right\|^{2}\right]$$
$$+\frac{2C_{\mathcal{K}}}{N^{2}h^{d-1}}\int\left\|\pi_{2}\left(\mathbf{u}\right)\right\|^{2}f_{\mathbf{U}}^{2}\left(\mathbf{u}\right)d\mathbf{u}$$
$$+O\left(N^{-2}\right)+o\left(N^{-2}h^{-(d-1)}+N^{-1}h^{P}\right),\text{ as }N\rightarrow\infty,$$

Also, it follows from equations (B-4) and (B-3) in Lemma B-1, that

$$\left\|\frac{1}{N}\sum_{i=1}^{N}E\left[\zeta_{3i}\right]-\delta_{3}\right\|^{2}=\left\|q_{3}-\delta_{3}\right\|^{2}$$
$$=\left\|h^{P}\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}+\gamma_{WK}\left(h\right)\right\|^{2}$$
$$=h^{2P}\left\|\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\|^{2}$$
$$+o\left(h^{2P}\right),\text{ as }N\rightarrow\infty,\qquad\text{(A-25)}$$

and similarly $N^{-1}\parallel\sum_{i=1}^{N}E\left[\zeta_{2i}\right]-\delta_{2}\parallel^{2}=h^{2P}\parallel\int\pi_{2}\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}\parallel^{2}+o\left(h^{2P}\right)$. Finally, it follows from (A-17), (A-24) and (A-25),

$$E\left[\left\|\widehat{\delta}_{3}\left(h\right)-\delta_{3}\right\|^{2}\right]=h^{2P}\left\|\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\|^{2}+O\left(\frac{1}{N}\right)+O\left(\frac{1}{N^{2}h^{d}}\right)$$
$$+O\left(\frac{h^{P}}{N}+\frac{1}{N^{2}}\right)+o\left(\frac{h^{P}}{N}+\frac{1}{N^{2}}+\frac{1}{N^{2}h^{d}}+h^{2P}\right),\text{ as }N\rightarrow\infty.\qquad\text{(A-26)}$$

Similarly

$$E\left[\left\|\widehat{\delta}_{2}\left(h\right)-\delta_{2}\right\|^{2}\right]=h^{2P}\left\|\int\pi_{2}\left(\mathbf{u}\right)S_{K}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}\right\|^{2}+O\left(\frac{1}{N}\right)+O\left(\frac{1}{N^{2}h^{d-1}}\right)$$
$$+O\left(\frac{h^{P}}{N}+\frac{1}{N^{2}}\right)+o\left(\frac{h^{P}}{N}+\frac{1}{N^{2}}+\frac{1}{N^{2}h^{d-1}}+h^{2P}\right),\text{ as }N\rightarrow\infty.\quad\text{(A-27)}$$

Terms: $E\left[\|\delta_4(h)-\delta_4\|^2\right]$ **and** $E\left[\|\delta_5(h)-\delta_5\|^2\right]$

We have

$$
\begin{aligned}
E\left[\left\|\widehat{\delta}_4(h)-\delta_4\right\|^2\right] &= E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{4i}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)\,\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\|^2\right]+E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}(\zeta_{4i}-E[\zeta_{4i}])\right\|^2\right]\\
&\quad+\left\|\frac{1}{N}\sum_{i=1}^{N}E[\zeta_{4i}]-\delta_4\right\|^2, \text{ and}\\
E\left[\left\|\widehat{\delta}_5(h)-\delta_5\right\|^2\right] &= E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{5i}\widehat{f}^2_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\|^2\right]+E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}(\zeta_{5i}-E[\zeta_{5i}])\right\|^2\right]\\
&\quad+\left\|\frac{1}{N}\sum_{i=1}^{N}E[\zeta_{5i}]-\delta_5\right\|^2. \qquad\text{(A-28)}
\end{aligned}
$$

We only show the working for $E[\|\ \widehat{\delta}_5(h)-\delta_5\ \|^2]$, the dominant term. We also work with the following expression

$$
\begin{aligned}
&E\left[\left\|\sum_{t=2}^{N}\sum_{s=2}^{N}W_h(V_t-v)\,\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\,W_h(V_s-v)\,\mathcal{K}_h(\mathbf{U}_s-\mathbf{u})\right\|^2\right]\\
&=(N-1)(N-2)(N-3)(N-4)\,\|E[W_h(V_2-v)\,\mathcal{K}_h(\mathbf{U}_2-\mathbf{u})]\|^4\\
&+6(N-1)(N-2)(N-3)\,E\left[\|W_h(V_2-v)\,\mathcal{K}_h(\mathbf{U}_2-\mathbf{u})\|^2\right]\|E[W_h(V_3-v)\,\mathcal{K}_h(\mathbf{U}_3-\mathbf{u})]\|^2\\
&+3(N-1)(N-2)\,E\left[\|W_h(V_2-v)\,\mathcal{K}_h(\mathbf{U}_2-\mathbf{u})\|^2\right]E\left[\|W_h(V_3-v)\,\mathcal{K}_h(\mathbf{U}_3-\mathbf{u})\|^2\right]\\
&+4(N-1)(N-2)\,E\left[W_h^3(V_2-v)\,\mathcal{K}_h^3(\mathbf{U}_2-\mathbf{u})\right]E[W_h(V_3-v)\,\mathcal{K}_h(\mathbf{U}_3-\mathbf{u})]\\
&+(N-1)\,E\left[\|W_h(V_2-v)\,\mathcal{K}_h(\mathbf{U}_2-\mathbf{u})\|^4\right].
\end{aligned}
$$

Firstly, Lemmas B-1 and B-2, imply that

$$
\begin{aligned}
&E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{5i}\widehat{f}^2_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\|^2\right]\\
&=\frac{1}{N(N-1)^4}\int\sigma_5^2(v,\mathbf{u})\,E\left[\left\|\sum_{t=2}^{N}W_h(V_t-v)\,\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\right\|^4\right]f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u} \qquad\text{(A-29)}\\
&=O\left(N^{-1}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-3}h^{-2d}\right)+O\left(N^{-4}h^{-3d}\right)\\
&+O\left(N^{-1}h^{4P}\right)+o\left(N^{-1}+N^{-2}h^{-d}\right).
\end{aligned}
$$

We now analyze the next term,

$$
E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}(\zeta_{5i}-E[\zeta_{5i}])\right\|^2\right]=\frac{1}{N}E\left[\|\zeta_{51}\|^2\right]+\frac{N-1}{N}E[\langle\zeta_{51},\zeta_{52}\rangle]-\|E[\zeta_{51}]\|^2.
$$

The first term in the last equation is like (A-29), after replacing $\sigma_5^2(v, \mathbf{u})$ by $\|\pi_5(v, \mathbf{u})\|^2$, and therefore, it is of the same order of magnitude. Now

$$\frac{N-1}{N} E\left[\langle \zeta_{51}, \zeta_{52} \rangle\right]$$

$$= \frac{1}{N(N-1)^3} E\left[\left\langle \pi_5(V_1, \mathbf{U}_1) \sum_{\substack{t=1 \\ t\neq 1, s\neq 1}}^{N} \sum_{s=1}^{N} W_h(V_t - V_1) \mathcal{K}_h(\mathbf{U}_t - \mathbf{U}_1) W_h(V_s - V_1) \mathcal{K}_h(\mathbf{U}_s - \mathbf{U}_1)\right.\right.$$

$$\left.\left., \pi_5(V_2, \mathbf{U}_2) \sum_{\substack{t=1 \\ t\neq 2, s\neq 2}}^{N} \sum_{s=1}^{N} W_h(V_t - V_2) \mathcal{K}_h(\mathbf{U}_t - \mathbf{U}_2) W_h(V_s - V_2) \mathcal{K}_h(\mathbf{U}_s - \mathbf{U}_2) \right\rangle\right].$$

Let us introduce the following notation here

$$\begin{aligned} \pi_{5;t} &\equiv \pi_5(V_t, \mathbf{U}_t) \\ W_{h;t1}\mathcal{K}_{h;t1} &\equiv W_h(V_t - V_1) \mathcal{K}_h(\mathbf{U}_t - \mathbf{U}_1), \text{ and} \\ W_{h;t2}\mathcal{K}_{h;t2} &\equiv W_h(V_t - V_2) \mathcal{K}_h(\mathbf{U}_t - \mathbf{U}_2), \end{aligned}$$

then it follows

$$\frac{N-1}{N} E\left[\langle \zeta_{51}, \zeta_{52} \rangle\right] = \frac{1}{N(N-1)^3}\left[\mathcal{B}_{5,I} + \mathcal{B}_{5,II} + \mathcal{B}_{5,III} + \mathcal{B}_{5,IV} + \mathcal{B}_{5,V} + \mathcal{B}_{5,VI}\right] + \mathcal{B}_{5,IV}$$

where

$$\begin{aligned}
\mathcal{B}_{5,I} &= E\left[\langle \pi_{5;1}, \pi_{5;2} \rangle W_{h;12}^4 \mathcal{K}_{h;12}^4\right], \\
\mathcal{B}_{5,II} &= (N-2) E\left[\langle \pi_{5;1}, \pi_{5;2} \rangle W_{h;12}^3 \mathcal{K}_{h;12}^3 W_{h;32}\mathcal{K}_{h;32}\right], \\
\mathcal{B}_{5,III} &= (N-2) E\left[\langle \pi_{5;1}, \pi_{5;2} \rangle W_{h;12}^2 \mathcal{K}_{h;12}^2 W_{h;31}\mathcal{K}_{h;31} W_{h;32}\mathcal{K}_{h;32}\right], \\
\mathcal{B}_{5,IV} &= (N-2)(N-3) E\left[\langle \pi_{5;1}, \pi_{5;2} \rangle W_{h;12}^2 \mathcal{K}_{h;12}^2 W_{h;41}\mathcal{K}_{h;41} W_{h;32}\mathcal{K}_{h;32}\right], \\
\mathcal{B}_{5,V} &= E\left[\langle \pi_{5;1}, \pi_{5;2} \rangle W_{h;12}\mathcal{K}_{h;12} \times\right. \\
&\quad \left\{(N-2) E\left[W_{h;31}\mathcal{K}_{h;31} W_{h;32}^2 \mathcal{K}_{h;32}^2 \middle| (\mathbf{V}_1, \mathbf{U}_1), (\mathbf{V}_2, \mathbf{U}_2)\right] + \right. \\
&\quad + 2(N-2)(N-3) E\left[W_{h;31}\mathcal{K}_{h;31} W_{h;32}\mathcal{K}_{h;32} W_{h;42}\mathcal{K}_{h;42} \middle| (\mathbf{V}_1, \mathbf{U}_1), (\mathbf{V}_2, \mathbf{U}_2)\right] \\
&\quad + (N-2)(N-3) E\left[W_{h;41}\mathcal{K}_{h;41} W_{h;32}^2 \mathcal{K}_{h;32}^2 \middle| (\mathbf{V}_1, \mathbf{U}_1), (\mathbf{V}_2, \mathbf{U}_2)\right] \\
&\quad \left.\left. + (N-2)(N-3)(N-4) E\left[W_{h;31}\mathcal{K}_{h;31} W_{h;42}\mathcal{K}_{h;42} W_{h;52}\mathcal{K}_{h;52} \middle| (\mathbf{V}_1, \mathbf{U}_1), (\mathbf{V}_2, \mathbf{U}_2)\right]\right\}\right], \\
\mathcal{B}_{5,VI} &= E\left[\langle \pi_{5;1}, \pi_{5;2} \rangle W_{h;12}^2 \mathcal{K}_{h;12}^2 \times\right. \\
&\quad \left\{(N-2) E\left[W_{h;32}^2 \mathcal{K}_{h;32}^2 \middle| (\mathbf{V}_1, \mathbf{U}_1), (\mathbf{V}_2, \mathbf{U}_2)\right]\right. \\
&\quad \left.\left. + (N-2)(N-3) E\left[W_{h;32}\mathcal{K}_{h;32} W_{h;42}\mathcal{K}_{h;42} \middle| (\mathbf{V}_1, \mathbf{U}_1), (\mathbf{V}_2, \mathbf{U}_2)\right]\right\}\right], \text{ and} \\
\mathcal{B}_{5,VII} &= \frac{1}{N(N-1)^3} E\left[\left\langle \pi_{5;1} \left\| \textstyle\sum_{t=3}^{N} W_{h;t1}\mathcal{K}_{h;t1} \right\|^2, \pi_{5;2} \left\| \textstyle\sum_{s=3}^{N} W_{h;s2}\mathcal{K}_{h;s2} \right\|^2 \right\rangle\right].
\end{aligned}$$

Then by Lemmas B-7, and B-8, this term is simply

$$\frac{N-1}{N}E\left[\langle\zeta_{51},\zeta_{52}\rangle\right]=\left\|E\left[\zeta_{51}\right]\right\|^{2}+O\left(N^{-1}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-3}h^{-2d}\right)$$
$$+O\left(N^{-4}h^{-3d}\right)+O\left(N^{-1}h^{P}\right),$$

and conclude that

$$E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_{5i}-E\left[\zeta_{5i}\right]\right)\right\|^{2}\right]=O\left(N^{-1}\right)+O\left(N^{-2}\right)+O\left(N^{-2}h^{-d}\right)$$
$$+O\left(N^{-1}h^{P}\right)+o\left(N^{-1}+N^{-2}h^{-2d}\right).$$

We now turn our attention to (A-28). Firstly,

$$\begin{aligned}E\left[\zeta_{51}\right]=&\frac{1}{N-1}E\left[\pi_{5}\left(V_{1},\mathbf{U}_{1}\right)W_{h}^{2}\left(V_{2}-V_{1}\right)\mathcal{K}_{h}^{2}\left(\mathbf{V}_{2}-\mathbf{V}_{1}\right)\right]\\&+E\left[\pi_{5}\left(V_{1},\mathbf{U}_{1}\right)W_{h}\left(V_{2}-V_{1}\right)\mathcal{K}_{h}\left(\mathbf{V}_{2}-\mathbf{V}_{1}\right)W_{h}\left(V_{3}-V_{1}\right)\mathcal{K}_{h}\left(\mathbf{V}_{3}-\mathbf{V}_{1}\right)\right]\\&=\frac{C_{W\mathcal{K}}}{Nh^{d}}\int\pi_{5}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}^{2}\left(v,\mathbf{u}\right)dvd\mathbf{u}+2h^{P}\left[\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right]\\&+\delta_{5}+o\left(N^{-1}h^{-d}\right)+o\left(h^{P}\right),\text{ and conclude}\end{aligned}$$

$$\left\|E\left[\zeta_{51}\right]-\delta_{5}\right\|^{2}=\left\|\frac{C_{W\mathcal{K}}}{Nh^{d}}\int\pi_{3}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right.$$
$$\left.+2h^{P}\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\|^{2},$$

because of Lemmas B-5, and B-1, and noticing that

$$\begin{aligned}\pi_{5}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)&=E\left[\left.\frac{\omega f_{\mathbf{U}}\left(\mathbf{U}\right)}{f_{V\mathbf{U}}^{3}\left(V,\mathbf{U}\right)}\right|V=v,\mathbf{U}=\mathbf{u}\right]f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\\&=E\left[\left.\frac{\omega f_{\mathbf{U}}\left(\mathbf{U}\right)}{f_{V\mathbf{U}}^{2}\left(V,\mathbf{U}\right)}\right|V=v,\mathbf{U}=\mathbf{u}\right]\\&\equiv\pi_{3}\left(v,\mathbf{u}\right).\end{aligned}$$

In conclusion, we have that

$$E\left[\left\|\widehat{\delta}_{5}\left(h\right)-\delta_{5}\right\|^{2}\right]=\left\|\frac{C_{W\mathcal{K}}}{Nh^{d}}\int\pi_{3}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right.$$
$$\left.+2h^{P}\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\|^{2}$$
$$+O\left(N^{-1}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-1}h^{2P}+N^{-2}\right).$$

By the exact same argument, we could also show that

$$E\left[\left\|\widehat{\delta}_{4}\left(h\right)-\delta_{4}\right\|^{2}\right]=h^{2P}\left\|\int\pi_{2}\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)d\mathbf{u}+\int\pi_{3}\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\|^{2}$$
$$+O\left(N^{-2}h^{2(d-1)}\right)+O\left(N^{-1}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-1}h^{2P}+N^{-2}\right),$$

where the first term of the last equation follows from Lemma B-1.

Terms: $E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_2(h)-\delta_2\right\rangle\right]$ **and** $E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_3(h)-\delta_3\right\rangle\right]$

As it was previously done, we have

$$E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_2(h)-\delta_2\right\rangle\right]=E\left[\left\langle\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{1i},\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{2i}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)\right\rangle\right]$$
$$+E\left[\left\langle\frac{1}{N}\sum_{i=1}^{N}(\pi_{1i}-E[\pi_{1i}]),\frac{1}{N}\sum_{i=1}^{N}(\zeta_{2i}-E[\zeta_{2i}])\right\rangle\right],$$

$$E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_3(h)-\delta_3\right\rangle\right]=E\left[\left\langle\frac{1}{N}\sum_{i=1}^{N}\widetilde{\varepsilon}_{1i},\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{3i}\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right\rangle\right]\tag{A-30}$$
$$+E\left[\left\langle\frac{1}{N}\sum_{i=1}^{N}(\widetilde{\pi}_{1i}-E[\widetilde{\pi}_{1i}]),\frac{1}{N}\sum_{i=1}^{N}(\zeta_{3i}-E[\zeta_{3i}])\right\rangle\right].\tag{A-31}$$

We show the working for $E\langle\ \widehat{\delta}_1-\delta_1,\widehat{\delta}_3(h)-\delta_3\rangle$ only. Firstly, Lemma B-1 implies that (A-30) equals

$$\frac{1}{N^2}\sum_{i=1}^{N}E\left[\widetilde{\sigma}_{13}(v_i,\mathbf{u}_i)\widehat{f}_{V\mathbf{U}}(v_i,\mathbf{u}_i)\right]$$
$$=\frac{1}{N(N-1)}\int\widetilde{\sigma}_{13}(v,\mathbf{u})E\left[\sum_{t=2}^{N}W_h(V_t-v)\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\right]f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}$$
$$=\frac{1}{N}\int\widetilde{\sigma}_{13}(v,\mathbf{u})f_{V\mathbf{U}}^2(v,\mathbf{u})\,dvd\mathbf{u}+O\left(h^P/N\right)+o\left(h^P\right),\text{ as }N\to\infty,$$

where $E\left[\widetilde{\varepsilon}_{1i}^{\top}\varepsilon_{3j}\middle|(V_1,\mathbf{U}_1),\ldots,(V_N,\mathbf{U}_N)\right]=1(i=j)\,\widetilde{\sigma}_{13}(V_i,\mathbf{U}_i)$, from Assumption (A6). Also

$$E\left[\left\langle\frac{1}{N}\sum_{i=1}^{N}(\widetilde{\pi}_{1i}-E[\widetilde{\pi}_{1i}]),\frac{1}{N}\sum_{i=1}^{N}(\zeta_{3i}-E[\zeta_{3i}])\right\rangle\right]=\frac{1}{N}E[\langle\widetilde{\pi}_{11},\zeta_{31}\rangle]$$
$$+\frac{N-1}{N}E[\langle\widetilde{\pi}_{11},\zeta_{32}\rangle]-\langle E[\widetilde{\pi}_{11}],E[\zeta_{31}]\rangle.$$

The first term equals, $N^{-1}E[\langle\widetilde{\pi}_{11},\zeta_{31}\rangle]$,

$$=\frac{1}{N(N-1)}\int\langle\widetilde{\pi}_1(v,\mathbf{u}),\pi_3(v,\mathbf{u})\rangle E\left[\sum_{t=2}^{N}W_h(V_t-v)\mathcal{K}_h(\mathbf{U}_t-\mathbf{u})\right]f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}$$
$$=\frac{1}{N}\int\langle\widetilde{\pi}_1(v,\mathbf{u}),\pi_3(v,\mathbf{u})\rangle f_{V\mathbf{U}}^2(v,\mathbf{u})\,dvd\mathbf{u}+O\left(h^P/N\right)+o\left(h^P\right),\text{ as }N\to\infty,$$

from Lemma B-1. The second term, $N^{-1}(N-1)E[\langle\pi_{11},\zeta_{32}\rangle]$, equals

$$\frac{1}{N}E\left[\langle\widetilde{\pi}_1(V_1,\mathbf{U}_1),\pi_3(V_2,\mathbf{U}_2)\rangle\sum_{\substack{t=1\\t\neq2}}^{N}W_h(V_t-V_2)\mathcal{K}_h(\mathbf{U}_t-\mathbf{U}_2)\right]$$
$$=\frac{1}{N}\mathcal{B}_{13,I}+\left(\frac{N-2}{N}\right)\mathcal{B}_{13,II},$$

where $\mathcal{B}_{13,I} = E\left[\left\langle \widetilde{\pi}_1 (V_1, \mathbf{U}_1), \pi_3 (V_2, \mathbf{U}_2)\right\rangle W_h (V_1 - V_2) \mathcal{K}_h (\mathbf{U}_1 - \mathbf{U}_2)\right]$ and $\mathcal{B}_{13,II} = E[\langle \pi_1 (V_1, \mathbf{U}_1), \pi_3 (V_2, \mathbf{U}_2) \rangle W_h (V_3 - V_2) \mathcal{K}_h (\mathbf{U}_3 - \mathbf{U}_2)]$. It follows from Lemma B-9 that

$$\frac{N-1}{N} E\left[\langle \pi_{11}, \zeta_{32}\rangle\right] = \frac{1}{N}\int \langle \widetilde{\pi}_1 (v, \mathbf{u}), \pi_3 (v, \mathbf{u})\rangle f_{V\mathbf{U}}^2 (v, \mathbf{u})\, dv d\mathbf{u} + \left(1 - \frac{2}{N}\right) \langle \delta_1, \delta_3\rangle$$
$$+ h^P \int \langle \delta_1, \pi_3 (v, \mathbf{u})\rangle S_{W\mathcal{K}} (v, \mathbf{u})\, dv d\mathbf{u}$$
$$+ o\left(\frac{h^P}{N} + \frac{1}{N}\right).$$

Finally $\langle E[\widetilde{\pi}_{11}], E[\zeta_{31}]\rangle = \langle \delta_1, \delta_3\rangle + h^P \int \langle \delta_1, \pi_3 (v, \mathbf{u})\rangle S_{W\mathcal{K}} (v, \mathbf{u})\, dv d\mathbf{u} + o\left(h^P\right)$. Therefore, we conclude

$$E\left[\left\langle \widehat{\delta}_1 - \delta_1, \widehat{\delta}_3 (h) - \delta_3\right\rangle\right] = O\left(\frac{1}{N}\right) + O\left(\frac{h^P}{N}\right) + o\left(\frac{h^P}{N} + \frac{1}{N}\right), \text{ and similarly,} \tag{A-32}$$
$$E\left[\left\langle \widehat{\delta}_1 - \delta_1, \widehat{\delta}_2 (h) - \delta_2\right\rangle\right] = O\left(\frac{1}{N}\right) + O\left(\frac{h^P}{N}\right) + o\left(\frac{h^P}{N} + \frac{1}{N}\right). \tag{A-33}$$

Terms: $E\left[\left\langle \widehat{\delta}_1 (h) - \delta_1, \widehat{\delta}_4 (h) - \delta_4\right\rangle\right]$ **and** $E\left[\left\langle \widehat{\delta}_1 (h) - \delta_1, \widehat{\delta}_5 (h) - \delta_5\right\rangle\right]$

As it was previously done, we have

$$E\left[\left\langle \widehat{\delta}_1 - \delta_1, \widehat{\delta}_4 (h) - \delta_4\right\rangle\right] = E\left[\left\langle \frac{1}{N}\sum_{i=1}^N \varepsilon_{1i}, \frac{1}{N}\sum_{i=1}^N \varepsilon_{4i} \widehat{f}_{V\mathbf{U}} (v_i, \mathbf{u}_i) \widehat{f}_{\mathbf{U}} (\mathbf{u}_i)\right\rangle\right]$$
$$+ E\left[\left\langle \frac{1}{N}\sum_{i=1}^N (\pi_{1i} - E[\pi_{1i}]), \frac{1}{N}\sum_{i=1}^N (\zeta_{4i} - E[\zeta_{4i}])\right\rangle\right],$$

$$E\left[\left\langle \widehat{\delta}_1 - \delta_1, \widehat{\delta}_5 (h) - \delta_5\right\rangle\right] = E\left[\left\langle \frac{1}{N}\sum_{i=1}^N \widetilde{\varepsilon}_{1i}, \frac{1}{N}\sum_{i=1}^N \varepsilon_{5i} \widehat{f}_{V\mathbf{U}}^2 (v_i, \mathbf{u}_i)\right\rangle\right] \tag{A-34}$$
$$+ E\left[\left\langle \frac{1}{N}\sum_{i=1}^N (\widetilde{\pi}_{1i} - E[\widetilde{\pi}_{1i}]), \frac{1}{N}\sum_{i=1}^N (\zeta_{5i} - E[\zeta_{5i}])\right\rangle\right]. \tag{A-35}$$

We show the working for $E\langle \widehat{\delta}_1 - \delta_1, \widehat{\delta}_5 (h) - \delta_5\rangle$ only. Firstly, Lemma B-2 implies that (A-34) equals

$$\frac{1}{N^2}\sum_{i=1}^N E\left[\widetilde{\sigma}_{15} (v_i, \mathbf{u}_i) \widehat{f}_{V\mathbf{U}}^2 (v_i, \mathbf{u}_i)\right]$$
$$= \frac{1}{N (N-1)^2}\int \widetilde{\sigma}_{15} (v, \mathbf{u}) E\left[\left\| \sum_{t=2}^N W_h (V_t - v) \mathcal{K}_h (\mathbf{U}_t - \mathbf{u})\right\|^2\right] f_{V\mathbf{U}} (v, \mathbf{u})\, dv d\mathbf{u}$$
$$= \frac{C_{W\mathcal{K}}}{N^2 h^d}\int \widetilde{\sigma}_{15} (v, \mathbf{u}) f_{V\mathbf{U}}^2 (v, \mathbf{u})\, dv d\mathbf{u} + \frac{1}{N}\int \widetilde{\sigma}_{15}^2 (v, \mathbf{u}) f_{V\mathbf{U}}^3 (v, \mathbf{u})\, dv d\mathbf{u}$$
$$+ O\left(h^P/N\right) + o\left(h^P\right) + o\left(N^{-2} h^{-d}\right), \text{ as } N \to \infty,$$

where $E\left[\widetilde{\varepsilon}_{1i}^{\top}\varepsilon_{5j}\middle|\left(V_1,\mathbf{U}_1\right),\ldots,\left(V_N,\mathbf{U}_N\right)\right]=1\left(i=j\right)\widetilde{\sigma}_{15}\left(V_i,\mathbf{U}_i\right)$, from Assumption (A6). Also

$$E\left[\left\langle\frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\pi}_{1i}-E\left[\widetilde{\pi}_{1i}\right]\right),\frac{1}{N}\sum_{i=1}^{N}\left(\zeta_{5i}-E\left[\zeta_{5i}\right]\right)\right\rangle\right]=\frac{1}{N}E\left[\left\langle\widetilde{\pi}_{11},\zeta_{51}\right\rangle\right]$$
$$+\frac{N-1}{N}E\left[\left\langle\widetilde{\pi}_{11},\zeta_{52}\right\rangle\right]-\left\langle E\left[\widetilde{\pi}_{11}\right],E\left[\zeta_{51}\right]\right\rangle.$$

The first term of the last equality, $N^{-1}E\left[\left\langle\widetilde{\pi}_{11},\zeta_{51}\right\rangle\right]$, is

$$=\frac{1}{N\left(N-1\right)^2}\int\left\langle\widetilde{\pi}_1\left(v,\mathbf{u}\right),\pi_5\left(v,\mathbf{u}\right)\right\rangle E\left[\left\|\sum_{t=2}^{N}W_h\left(V_t-v\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)\right\|^2\right]f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}$$
$$=\frac{C_{W\mathcal{K}}}{N^2h^d}\int\left\langle\widetilde{\pi}_1\left(v,\mathbf{u}\right),\pi_5\left(v,\mathbf{u}\right)\right\rangle f_{V\mathbf{U}}^2\left(v,\mathbf{u}\right)dvd\mathbf{u}+\frac{1}{N}\int\left\langle\widetilde{\pi}_1\left(v,\mathbf{u}\right),\pi_5\left(v,\mathbf{u}\right)\right\rangle f_{V\mathbf{U}}^3\left(v,\mathbf{u}\right)dvd\mathbf{u}$$
$$+O\left(h^P/N\right)+o\left(h^P\right)+o\left(N^{-2}h^{-d}\right),\text{ as }N\to\infty\text{, from Lemma B-2}$$

The second term, $N^{-1}\left(N-1\right)E\left[\left\langle\widetilde{\pi}_{11},\zeta_{52}\right\rangle\right]$, equals

$$\frac{1}{N\left(N-1\right)}E\left[\left\langle\widetilde{\pi}_1\left(V_1,\mathbf{U}_1\right),\pi_5\left(V_2,\mathbf{U}_2\right)\right\rangle\left\|\sum_{\substack{t=1\\t\neq2}}^{N}W_h\left(V_t-V_2\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{U}_2\right)\right\|^2\right]$$
$$=\frac{1}{N\left(N-1\right)}\left[\left(N-2\right)\mathcal{B}_{15,I}+\left(N-2\right)\left(N-3\right)\mathcal{B}_{15,II}+2\left(N-2\right)\mathcal{B}_{15,III}+\mathcal{B}_{15,IV}\right]$$
$$=\frac{1}{N}\left[\mathcal{B}_{15,I}+2\mathcal{B}_{15,III}\right]+\mathcal{B}_{15,II}\left[1-\frac{5}{N}\right]+\frac{1}{N^2}\mathcal{B}_{15,IV}$$
$$+\frac{1}{N^2}\left[6\mathcal{B}_{15,II}-2\mathcal{B}_{15,I}-4\mathcal{B}_{15,III}\right]+o\left(N^{-2}\right),\text{ where}$$

$$\mathcal{B}_{15,I}=E\left[\left\langle\widetilde{\pi}_1\left(V_1,\mathbf{U}_1\right),\pi_5\left(V_2,\mathbf{U}_2\right)W_h^2\left(V_3-V_2\right)\mathcal{K}_h^2\left(\mathbf{U}_3-\mathbf{U}_2\right)\right\rangle\right]$$
$$\mathcal{B}_{15,II}=E\left[\left\langle\widetilde{\pi}_1\left(V_1,\mathbf{U}_1\right),\pi_5\left(V_2,\mathbf{U}_2\right)W_h\left(V_4-V_2\right)\mathcal{K}_h\left(\mathbf{U}_4-\mathbf{U}_2\right)W_h\left(V_3-V_2\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_2\right)\right\rangle\right]$$
$$\mathcal{B}_{15,III}=E\left[\left\langle\widetilde{\pi}_1\left(V_1,\mathbf{U}_1\right)W_h\left(V_1-V_2\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{U}_2\right),\pi_5\left(V_2,\mathbf{U}_2\right)W_h\left(V_3-V_2\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_2\right)\right\rangle\right]$$
$$\mathcal{B}_{15,IV}=E\left[\left\langle\widetilde{\pi}_1\left(V_1,\mathbf{U}_1\right),\pi_5\left(V_2,\mathbf{U}_2\right)W_h^2\left(V_1-V_2\right)\mathcal{K}_h^2\left(\mathbf{U}_1-\mathbf{U}_2\right)\right\rangle\right]$$

Finally, $\left\langle E\left[\widetilde{\pi}_{11}\right],E\left[\zeta_{51}\right]\right\rangle=N^{-1}\mathcal{B}_{15,I}+\mathcal{B}_{15,II}$, and conclude from Lemma B-10 that

$$E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_5\left(h\right)-\delta_5\right\rangle\right]=O\left(N^{-1}\right)+O\left(N^{-2}h^{-d}\right)$$
$$+O\left(\frac{h^P}{N}+\frac{1}{N^2}\right)+o\left(\frac{1}{N^2h^d}+\frac{h^P}{N}+\frac{1}{N^2}+h^{2P}\right),\text{ as }N\to\infty$$

Similarly, we infer from this result that

$$E\left[\left\langle\widehat{\delta}_1-\delta_1,\widehat{\delta}_4\left(h\right)-\delta_4\right\rangle\right]=O\left(N^{-1}\right)+O\left(N^{-2}h^{-(d-1)}\right)$$
$$+O\left(\frac{h^P}{N}+\frac{1}{N^2}\right)+o\left(\frac{1}{N^2h^{(d-1)}}+\frac{h^P}{N}+\frac{1}{N^2}+h^{2P}\right),\text{ as }N\to\infty$$

and therefore of smaller order.

Terms: $E\left[\left\langle \widehat{\delta}_2(h)-\delta_2, \widehat{\delta}_3(h)-\delta_3\right\rangle\right]$

We have

$$E\left[\left\langle \widehat{\delta}_2(h)-\delta_2, \widehat{\delta}_3(h)-\delta_3\right\rangle\right] = E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\widetilde{\varepsilon}_{2i}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i), \frac{1}{N}\sum_{j=1}^{N}\varepsilon_{3j}\widehat{f}_{V\mathbf{U}}(v_j,\mathbf{u}_j)\right\rangle\right] \quad \text{(A-36)}$$

$$+ E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right), \frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{3j}-E\left[\zeta_{31}\right]\right)\right\rangle\right] \quad \text{(A-37)}$$

$$+ \left\langle E[\widetilde{\zeta}_{21}]-\delta_2, E\left[\zeta_{31}\right]-\delta_3\right\rangle, \quad \text{(A-38)}$$

where $\widetilde{\zeta}_{2i} \equiv \widetilde{\pi}_2(v_i,\mathbf{u}_i)\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)$, and by construction $E\left[\widetilde{\varepsilon}_{2i}|\left(V_1,\mathbf{U}_1\right),\ldots,\left(V_N,\mathbf{U}_N\right)\right]=0$. As before

$$E\left[\widetilde{\varepsilon}_{2i}^{\top}\varepsilon_{3j}\middle|\left(V_1,\mathbf{U}_1\right),\ldots,\left(V_N,\mathbf{U}_N\right)\right] = \begin{cases} \widetilde{\sigma}_{23}(V_i,\mathbf{U}_i), & i=j, \\ 0, & i\neq j. \end{cases}$$

By Assumption (A6), the term (A-36) is equal to

$$\frac{1}{N(N-1)^2}\int\widetilde{\sigma}_{23}(v,\mathbf{u})\,E\left\langle\sum_{t=2}^{N}W_h(V_t-v)\,\mathcal{K}_h(\mathbf{U}_t-\mathbf{u}), \sum_{s=2}^{N}\mathcal{K}_h(\mathbf{U}_s-\mathbf{u})\right\rangle f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u},$$

where from Lemmas B-1, B-2 and B-4, it follows that

$$\begin{aligned}
&E\left\langle\sum_{t=2}^{N}W_h(V_t-v)\,\mathcal{K}_h(\mathbf{U}_t-\mathbf{u}), \sum_{s=2}^{N}\mathcal{K}_h(\mathbf{U}_s-\mathbf{u})\right\rangle \\
&= (N-1)\,E\left[W_h(V_1-v)\left\|\mathcal{K}_h(\mathbf{U}_1-\mathbf{u})\right\|^2\right] \\
&+ (N-1)(N-2)\,E\left[W_h(V_1-v)\,\mathcal{K}_h(\mathbf{U}_1-\mathbf{u})\right]E\left[\mathcal{K}_h(\mathbf{U}_1-\mathbf{u})\right] \\
&= (N-1)\left[C_{\mathcal{K}}f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-(d-1)}+\psi^{*}_{W\mathcal{K}}(h,(v,\mathbf{u}))\right] \\
&+ (N-1)(N-2)\left[f_{V\mathbf{U}}(v,\mathbf{u})+h^{P}S_{W\mathcal{K}}(v,\mathbf{u})+\beta_{W\mathcal{K}}(h,(v,\mathbf{u}))\right]\times \\
&\qquad \left[f_{\mathbf{U}}(\mathbf{u})+h^{P}S_{\mathcal{K}}(\mathbf{u})+\beta_{\mathcal{K}}(h,\mathbf{u})\right].
\end{aligned}$$

Therefore, we have that the term (A-36) is equal to

$$\frac{1}{N}\int\widetilde{\sigma}_{23}(v,\mathbf{u})f^2_{v\mathbf{u}}(v,\mathbf{u})\,f_{\mathbf{U}}(\mathbf{u})\,dvd\mathbf{u}+\frac{C_{\mathcal{K}}}{N^2h^{d-1}}\int\widetilde{\sigma}_{23}(v,\mathbf{u})\,f^2_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}$$
$$+\,O\left(\frac{h^P}{N}+\frac{1}{N^2}\right)+o\left(\frac{1}{N^2h^{d-1}}\right), \text{ as } N\to\infty.$$

Let us turn our attention to term (A-37),

$$E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right), \frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{3j}-E\left[\zeta_{31}\right]\right)\right\rangle\right]$$
$$= \frac{1}{N^2}\sum_{i=1}^{N}E\left[\left\langle\widetilde{\zeta}_{2i},\zeta_{3i}\right\rangle\right]+\frac{1}{N^2}\sum_{\substack{i=1\\ i\neq j}}^{N}\sum_{j=1}^{N}E\left[\left\langle\widetilde{\zeta}_{2i},\zeta_{3j}\right\rangle\right]-\left\langle E[\widetilde{\zeta}_{21}],E\left[\zeta_{31}\right]\right\rangle$$
$$= \frac{1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{31}\right\rangle\right]+\frac{N-1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{32}\right\rangle\right]-\left\langle q_2,q_3\right\rangle.$$

Also $E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{31}\right\rangle\right]$ equals

$$\frac{1}{N^2}\int\left\langle\widetilde{\pi}_2\left(v,\mathbf{u}\right),\pi_3\left(v,\mathbf{u}\right)\right\rangle E\left[\sum_{t=2}^{N}\sum_{s=2}^{N}K_h\left(\mathbf{U}_t-\mathbf{u}\right)W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right]f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u},$$

where $E\left[\sum_{t=2}^{N}\sum_{s=2}^{N}\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right] = (N-1)\,E\left[W_h\left(V_1-v\right)\times \left\|\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right\|^2\right]+(N-1)\left(N-2\right)E\left[\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]E\left[W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]$. It follows from Lemmas B-1, B-2 and B-4 that

$$\begin{aligned}&E\left[\sum_{t=2}^{N}\sum_{s=2}^{N}\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{u}\right)W_h\left(V_s-v\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{u}\right)\right]\\
&=(N-1)\left(C_{\mathcal{K}}f_{V\mathbf{U}}\left(v,\mathbf{u}\right)h^{-(d-1)}+\psi^*_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right)\\
&+(N-1)\left(N-2\right)\left[f_{\mathbf{U}}\left(\mathbf{u}\right)+h^PS_{\mathcal{K}}\left(\mathbf{u}\right)+\beta_{\mathcal{K}}\left(h,\mathbf{u}\right)\right]\times\\
&\qquad\left[f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^PS_{W\mathcal{K}}\left(v,\mathbf{u}\right)+\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{31}\right\rangle\right]&=\frac{1}{N}\int\left\langle\widetilde{\pi}_2\left(v,\mathbf{u}\right),\pi_3\left(v,\mathbf{u}\right)\right\rangle f^2_{V\mathbf{U}}\left(v,\mathbf{u}\right)f_{\mathbf{u}}\left(\mathbf{u}\right)dvd\mathbf{u}\\
&+\frac{1}{N^2h^{d-1}}C_{\mathcal{K}}\int\left\langle\widetilde{\pi}_2\left(v,\mathbf{u}\right),\pi_3\left(v,\mathbf{u}\right)\right\rangle f^2_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\\
&+O\left(N^{-1}h^P+N^{-2}\right)+o\left(N^{-2}h^{-(d-1)}\right),\text{ as } N\to\infty.\end{aligned}$$

We now turn our attention to

$$\frac{N-1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{32}\right\rangle\right]=N^{-3}\left(N-1\right)\sum_{\substack{t=1\\ t\neq 1,s\neq 2}}^{N}\sum_{s=1}^{N}\Delta_{23,ts},$$

where $\Delta_{23,ts}=E[\langle\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_t-\mathbf{U}_1\right),\pi_3\left(V_2,\mathbf{U}_2\right)W_h\left(V_s-V_2\right)\mathcal{K}_h\left(\mathbf{U}_s-\mathbf{U}_2\right)\rangle]$. Furthermore, we write

$$\Delta_{23,ts}=\begin{cases}\mathcal{B}_{23,I} & ; \quad s=t,\\ \mathcal{B}_{23,II} & ; \quad s\neq t,\ t\neq 2,\ s\neq 1,\\ \mathcal{B}_{23,III} & ; \quad s=t\ \&\ t=2,\ s\neq 1 \text{ or } t\neq 2,\ s=1,\\ \mathcal{B}_{23,IV} & ; \quad s\neq t,\ t=2,\ s=1.\end{cases}$$

Here we make the following definitions:

$$
\begin{aligned}
\mathcal{B}_{23,I} &= E\left[\left\langle E\left[\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\right|\mathbf{U}_3\right],\\
&\qquad E\left[\pi_3\left(V_1,\mathbf{U}_1\right)W_h\left(V_3-V_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\right|V_3,\mathbf{U}_3\right]\rangle] \qquad ;\\
\mathcal{B}_{23,II} &= \left\langle E\left[\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\right],\right.\\
&\qquad \left.E\left[\pi_3\left(V_1,\mathbf{U}_1\right)W_h\left(V_3-V_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\right]\right\rangle \equiv \langle q_2,q_3\rangle \qquad ;\\
\mathcal{B}_{23,III} &= E\left[\left\langle\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right),\pi_3\left(V_2,\mathbf{U}_2\right)\right\rangle\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{U}_1\right)W_h\left(V_3-V_2\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_2\right)\right] \qquad ;\\
\mathcal{B}_{23,IV} &= E\left[\left\langle\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right),\pi_3\left(V_2,\mathbf{U}_2\right)\right\rangle W_h\left(V_1-V_2\right)\left\|\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{U}_2\right)\right\|^2\right].
\end{aligned}
$$

Therefore, we are able to write

$$
\begin{aligned}
\frac{N-1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{32}\right\rangle\right] &= \frac{1}{N}\left[\mathcal{B}_{23,I}+2\mathcal{B}_{23,III}\right]+\langle q_2,q_3\rangle\left[1-\frac{5}{N}\right]+\frac{1}{N^2}\mathcal{B}_{23,IV}\\
&\quad+\frac{1}{N^2}\left[6\left\langle\widetilde{q}_2,q_3\right\rangle-2\mathcal{B}_{23,I}-4\mathcal{B}_{23,III}\right]+o\left(N^{-2}\right).
\end{aligned}
$$

Finally, term (A-38) is such that

$$
\begin{aligned}
&\left\langle E[\widetilde{\zeta}_{21}]-\delta_2\,,\,E\left[\zeta_{31}\right]-\delta_3\right\rangle = \langle q_2-\delta_2,q_3-\delta_3\rangle\\
&\qquad = h^{2P}\left\langle\int\pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u},\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\rangle\\
&\qquad +o\left(h^{2P}\right),\text{ as }N\to\infty,
\end{aligned}
$$

which follows from Lemma B-5.

Combining these pieces together, we obtain

$$
\begin{aligned}
&E\left[\left\langle\widehat{\delta}_2\left(h\right)-\delta_2,\widehat{\delta}_3\left(h\right)-\delta_3\right\rangle\right]\\
&= h^{2P}\left\langle\int\pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u},\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\rangle\\
&+O\left(N^{-1}\right)+O\left(N^2h^{d-1}\right)+O\left(\frac{h^P}{N}+\frac{1}{N^2}\right)\\
&+o\left(\frac{h^P}{N}+\frac{1}{N^2}+h^{2P}\right),\text{ as }N\to\infty, \qquad \text{(A-39)}
\end{aligned}
$$

Terms: $E\left[\left\langle \widehat{\delta}_2(h)-\delta_2, \widehat{\delta}_4(h)-\delta_4 \right\rangle\right]$ **and** $E\left[\left\langle \widehat{\delta}_2(h)-\delta_2, \widehat{\delta}_5(h)-\delta_5 \right\rangle\right]$

As before,

$$E\left[\left\langle \widehat{\delta}_2(h)-\delta_2, \widehat{\delta}_4(h)-\delta_4 \right\rangle\right] = E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\widetilde{\varepsilon}_{2i}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i), \frac{1}{N}\sum_{j=1}^{N}\varepsilon_{4j}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)\widehat{f}_{V\mathbf{U}}(v_j,\mathbf{u}_j) \right\rangle\right] \tag{A-40}$$

$$+ E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right), \frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{4j}-E[\zeta_{41}]\right) \right\rangle\right] \tag{A-41}$$

$$+ \left\langle E[\widetilde{\zeta}_{21}]-\delta_2, E[\zeta_{41}]-\delta_4 \right\rangle, \text{ similarly} \tag{A-42}$$

$$E\left[\left\langle \widehat{\delta}_2(h)-\delta_2, \widehat{\delta}_5(h)-\delta_5 \right\rangle\right] = E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\widetilde{\varepsilon}_{2i}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i), \frac{1}{N}\sum_{j=1}^{N}\varepsilon_{5j}\widehat{f}^{2}_{V\mathbf{U}}(v_j,\mathbf{u}_j) \right\rangle\right] \tag{A-43}$$

$$+ E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right), \frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{5j}-E[\zeta_{51}]\right) \right\rangle\right] \tag{A-44}$$

$$+ \left\langle E[\widetilde{\zeta}_{21}]-\delta_2, E[\zeta_{51}]-\delta_5 \right\rangle. \tag{A-45}$$

We also notice that

$$E\left[\widetilde{\varepsilon}_{2i}^{\top}\varepsilon_{4j}\middle|(V_1,\mathbf{U}_1),\ldots,(V_N,\mathbf{U}_N)\right] = \begin{cases} \widetilde{\sigma}_{24}(V_i,\mathbf{U}_i), & i=j, \\ 0, & i\neq j. \end{cases}$$

$$E\left[\widetilde{\varepsilon}_{2i}^{\top}\varepsilon_{5j}\middle|(V_1,\mathbf{U}_1),\ldots,(V_N,\mathbf{U}_N)\right] = \begin{cases} \widetilde{\sigma}_{25}(V_i,\mathbf{U}_i), & i=j, \\ 0, & i\neq j. \end{cases}$$

In what follows, we also use the following quantities:

$$E\left\langle \sum_{i=2}^{N}\mathcal{K}_h\left(\mathbf{U}_i-\mathbf{u}\right),\sum_{j=2}^{N}\sum_{k=2}^{N}\mathcal{K}_h\left(\mathbf{U}_j-\mathbf{u}\right)W_h\left(V_k-v\right)\mathcal{K}_h\left(\mathbf{U}_k-\mathbf{u}\right)\right\rangle \tag{A-46}$$
$$=\left(N-1\right)E\left[W_h\left(V_2-v\right)\mathcal{K}_h^3\left(\mathbf{U}_2-\mathbf{u}\right)\right]$$
$$+\left(N-1\right)\left(N-2\right)E\left[W_h\left(V_3-v\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{u}\right)\right]E\left[\mathcal{K}_h^2\left(\mathbf{U}_2-\mathbf{u}\right)\right]$$
$$+2\left(N-1\right)\left(N-2\right)E\left[W_h\left(V_2-v\right)\mathcal{K}_h^2\left(\mathbf{U}_2-\mathbf{u}\right)\right]E\left[\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{u}\right)\right]$$
$$+\left(N-1\right)\left(N-2\right)\left(N-3\right)\left\|E\left[\mathcal{K}_h\left(\mathbf{U}_4-\mathbf{u}\right)\right]\right\|^2E\left[W_h\left(V_3-v\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{u}\right)\right]$$

$$E\left\langle \sum_{i=2}^{N}\mathcal{K}_h\left(\mathbf{U}_i-\mathbf{u}\right),\sum_{j=2}^{N}\sum_{k=2}^{N}W_h\left(V_j-v\right)\mathcal{K}_h\left(\mathbf{U}_j-\mathbf{u}\right)W_h\left(V_k-v\right)\mathcal{K}_h\left(\mathbf{U}_k-\mathbf{u}\right)\right\rangle \tag{A-47}$$
$$=\left(N-1\right)E\left[W_h^2\left(V_2-v\right)\mathcal{K}_h^3\left(\mathbf{U}_2-\mathbf{u}\right)\right]$$
$$+2\left(N-1\right)\left(N-2\right)E\left[W_h\left(V_3-v\right)\mathcal{K}_h^2\left(\mathbf{U}_3-\mathbf{u}\right)\right]E\left[W_h\left(V_2-v\right)\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{u}\right)\right]$$
$$+\left(N-1\right)\left(N-2\right)E\left[\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{u}\right)\right]E\left[W_h^2\left(V_3-v\right)\mathcal{K}_h^2\left(\mathbf{U}_3-\mathbf{u}\right)\right]$$
$$+\left(N-1\right)\left(N-2\right)\left(N-3\right)E\left[\mathcal{K}_h\left(\mathbf{U}_4-\mathbf{u}\right)\right]\left\|E\left[W_h\left(V_3-v\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{u}\right)\right]\right\|^2.$$

By Assumption (A6), the terms (A-40) and (A-43) are

$$\frac{1}{N\left(N-1\right)^3}\int\widetilde{\sigma}_{24}\left(v,\mathbf{u}\right)$$
$$\times E\left\langle \sum_{i=2}^{N}\mathcal{K}_h\left(\mathbf{U}_i-\mathbf{u}\right),\sum_{j=2}^{N}\sum_{k=2}^{N}\mathcal{K}_h\left(\mathbf{U}_j-\mathbf{u}\right)W_h\left(V_k-v\right)\mathcal{K}_h\left(\mathbf{U}_k-\mathbf{u}\right)\right\rangle f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u},\text{ and} \tag{A-48}$$

$$\frac{1}{N\left(N-1\right)^3}\int\widetilde{\sigma}_{25}\left(v,\mathbf{u}\right)$$
$$\times E\left\langle \sum_{i=2}^{N}\mathcal{K}_h\left(\mathbf{U}_i-\mathbf{u}\right),\sum_{j=2}^{N}\sum_{k=2}^{N}W_h\left(V_j-v\right)\mathcal{K}_h\left(\mathbf{U}_j-\mathbf{u}\right)W_h\left(V_k-v\right)\mathcal{K}_h\left(\mathbf{U}_k-\mathbf{u}\right)\right\rangle \tag{A-49}$$
$$\times f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}.$$

After plugging (A-46) and (A-47) in (A-48) and (A-49) respectively, and using Lemmas B-1, B-2, and B-4, it follows that

$$E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\widetilde{\varepsilon}_{2i}\widehat{f}_{\mathbf{U}}\left(\mathbf{u}_i\right),\frac{1}{N}\sum_{j=1}^{N}\varepsilon_{4j}\widehat{f}_{\mathbf{U}}\left(\mathbf{u}_i\right)\widehat{f}_{V\mathbf{U}}\left(v_j,\mathbf{u}_j\right)\right\rangle\right]$$
$$=O\left(N^{-1}\right)+O\left(N^{-3}h^{-2(d-1)}\right)+O\left(N^{-2}h^{-(d-1)}\right)+O\left(N^{-1}h^{P}\right),\text{ and}$$
$$E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\widetilde{\varepsilon}_{2i}\widehat{f}_{\mathbf{U}}\left(\mathbf{u}_i\right),\frac{1}{N}\sum_{j=1}^{N}\varepsilon_{5j}\widehat{f}_{V\mathbf{U}}^2\left(v_j,\mathbf{u}_j\right)\right\rangle\right]$$
$$=O\left(N^{-1}\right)+O\left(N^{-3}h^{-(2d-1)}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-2}h^{-(d-1)}\right)+O\left(N^{-1}h^{P}\right).$$

Now, terms (A-41) and (A-44) can be written as

$$E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right),\frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{4j}-E\left[\zeta_{41}\right]\right)\right\rangle\right]$$
$$=\frac{1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{41}\right\rangle\right]+\frac{N-1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{42}\right\rangle\right]-\left\langle E[\widetilde{\zeta}_{21}],E\left[\zeta_{41}\right]\right\rangle,\text{ and}$$
$$E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right),\frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{5j}-E\left[\zeta_{51}\right]\right)\right\rangle\right]$$
$$=\frac{1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{51}\right\rangle\right]+\frac{N-1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{52}\right\rangle\right]-\left\langle E[\widetilde{\zeta}_{21}],E\left[\zeta_{51}\right]\right\rangle\text{ respectively.}$$

The terms $N^{-1}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{41}\right\rangle\right]$ and $N^{-1}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{51}\right\rangle\right]$ are like (A-48) and (A-49) after replacing $\widetilde{\sigma}_{24}\left(v,\mathbf{u}\right)$ and $\widetilde{\sigma}_{25}\left(v,\mathbf{u}\right)$ by $\left\langle\widetilde{\pi}_2\left(v,\mathbf{u}\right),\pi_4\left(v,\mathbf{u}\right)\right\rangle$, and $\left\langle\widetilde{\pi}_2\left(v,\mathbf{u}\right),\pi_5\left(v,\mathbf{u}\right)\right\rangle$ respectively. Therefore, they are of the same order. In particular,

$$\frac{1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{41}\right\rangle\right]$$
$$=O\left(N^{-1}\right)+O\left(N^{-3}h^{-2(d-1)}\right)+O\left(N^{-2}h^{-(d-1)}\right)+O\left(N^{-1}h^{P}\right),\text{ and}$$
$$\frac{1}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{51}\right\rangle\right]$$
$$=O\left(N^{-1}\right)+O\left(N^{-3}h^{-(2d-1)}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-2}h^{-(d-1)}\right)+O\left(N^{-1}h^{P}\right).$$

We only show the working for $N^{-1}\left(N-1\right)E[\langle\ \widetilde{\zeta}_{21},\zeta_{52}\ \rangle]$, which is the leading term, in any case:

$$\frac{(N-1)}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{52}\right\rangle\right]$$
$$=\frac{1}{N\left(N-1\right)^2}\left[\mathcal{B}_{25,I}+\mathcal{B}_{25,II}+\mathcal{B}_{25,III}+\mathcal{B}_{25,IV}+\mathcal{B}_{25,V}+\mathcal{B}_{25,VI}\right]+\mathcal{B}_{25,IV},$$

where

$$\mathcal{B}_{25,I}=E\left[\left\langle\widetilde{\pi}_{2;1},\pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}^3\right],$$
$$\mathcal{B}_{25,II}=\left(N-2\right)E\left[\left\langle\widetilde{\pi}_{2;1},\pi_{5;2}\right\rangle\mathcal{K}_{h;12}^2W_{h;32}\mathcal{K}_{h;32}\right],$$
$$\mathcal{B}_{25,III}=2\left(N-2\right)E\left[\left\langle\widetilde{\pi}_{2;1},\pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}^2\mathcal{K}_{h;32}\right],$$
$$\mathcal{B}_{25,IV}=E\left[\left\langle\widetilde{\pi}_{2;1},\pi_{5;2}\right\rangle\mathcal{K}_{h;12}\left\{\left(N-2\right)W_{h;32}\mathcal{K}_{h;32}^3+\left(N-2\right)\left(N-3\right)\mathcal{K}_{h;32}W_{h;42}\mathcal{K}_{h;42}\right\}\right],$$
$$\mathcal{B}_{25,V}=E\left[\left\langle\widetilde{\pi}_{2;1},\pi_{5;2}\right\rangle\mathcal{K}_{h;12}\left\{\left(N-2\right)\mathcal{K}_{h;31}W_{h;32}\mathcal{K}_{h;32}+\left(N-2\right)\left(N-3\right)\mathcal{K}_{h;31}W_{h;42}\mathcal{K}_{h;42}\right\}\right],$$
$$\mathcal{B}_{25,VI}=E\left[\left\langle\widetilde{\pi}_{2;1},\pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}\left\{\left(N-2\right)\mathcal{K}_{h;31}\mathcal{K}_{h;32}+\left(N-2\right)\left(N-3\right)\mathcal{K}_{h;31}\mathcal{K}_{h;42}\right\}\right],$$
$$\mathcal{B}_{25,VII}=\frac{1}{N\left(N-1\right)^2}E\left[\left\langle\widetilde{\pi}_{2;1}\textstyle\sum_{t=3}^{N}\mathcal{K}_{h;t1},\pi_{5;2}\left\|\sum_{t=3}^{N}W_{h;t2}\mathcal{K}_{h;t2}\right\|^2\right\rangle\right].$$

It then follows from Lemmas B-1, B-12 and B-13, that

$$\frac{(N-1)}{N}E\left[\left\langle\widetilde{\zeta}_{21},\zeta_{52}\right\rangle\right]=\left\langle E[\widetilde{\zeta}_{21}],E\left[\zeta_{51}\right]\right\rangle+O\left(N^{-1}\right)$$
$$+O\left(N^{-2}h^{-2(d-1)}\right)+O\left(N^{-1}h^{P}\right)$$

and conclude

$$E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\left(\widetilde{\zeta}_{2i}-E[\widetilde{\zeta}_{21}]\right), \frac{1}{N}\sum_{j=1}^{N}\left(\zeta_{4j}-E\left[\zeta_{41}\right]\right)\right\rangle\right]$$
$$=O\left(N^{-1}\right)+O\left(N^{-2}h^{-2(d-1)}\right)+O\left(N^{-2}h^{-d}\right)+O\left(N^{-2}h^{-(d-1)}\right)+O\left(N^{-1}h^{P}\right).$$

Likewise, (A-41) will be of smaller order than this term and therefore, it will not contribute towards the leading terms in the expansion. The only contributions will be from terms (A-42), and (A-45). In particular,

$$\left\langle E[\widetilde{\zeta}_{21}]-\delta_2, E\left[\zeta_{41}\right]-\delta_4\right\rangle$$
$$=h^{2P}\left\langle\int \pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}\right.$$
$$\left., \int \pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}+\int \pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\rangle+o\left(N^{-1}h^{P-d}\right),$$
$$\left\langle E[\widetilde{\zeta}_{21}]-\delta_2, E\left[\zeta_{51}\right]-\delta_5\right\rangle$$
$$=2h^{2P}\left\langle\int \pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}, \int \pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\rangle$$
$$+N^{-1}h^{P-d}\left\langle\int \pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}, C_{W\mathcal{K}}\int \pi_3\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right\rangle.$$

by Lemma B-1.

Terms: $E\left[\left\langle \widehat{\delta}_3(h)-\delta_3,\widehat{\delta}_4(h)-\delta_4\right\rangle\right]$**,** $E\left[\left\langle \widehat{\delta}_3(h)-\delta_3,\widehat{\delta}_5(h)-\delta_5\right\rangle\right]$

and $E\left[\left\langle \widehat{\delta}_4(h)-\delta_4,\widehat{\delta}_5(h)-\delta_5\right\rangle\right]$

As before,

$$E\left[\left\langle \widehat{\delta}_3(h)-\delta_3,\widehat{\delta}_4(h)-\delta_4\right\rangle\right]=E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{3i}\widehat{f}_{V\mathbf{U}}(v,\mathbf{u}_i),\frac{1}{N}\sum_{j=1}^{N}\varepsilon_{4j}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)\,\widehat{f}_{V\mathbf{U}}(v_j,\mathbf{u}_j)\right\rangle\right] \tag{A-50}$$

$$+E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}(\zeta_{3i}-E[\zeta_{31}]),\frac{1}{N}\sum_{j=1}^{N}(\zeta_{4j}-E[\zeta_{41}])\right\rangle\right] \tag{A-51}$$

$$+\langle E[\zeta_{31}]-\delta_3,E[\zeta_{41}]-\delta_4\rangle, \tag{A-52}$$

$$E\left[\left\langle \widehat{\delta}_3(h)-\delta_3,\widehat{\delta}_5(h)-\delta_5\right\rangle\right]=E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{3i}\widehat{f}_{V\mathbf{U}}(v,\mathbf{u}_i),\frac{1}{N}\sum_{j=1}^{N}\varepsilon_{5j}\widehat{f}^2_{V\mathbf{U}}(v_j,\mathbf{u}_j)\right\rangle\right] \tag{A-53}$$

$$+E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}(\zeta_{3i}-E[\zeta_{31}]),\frac{1}{N}\sum_{j=1}^{N}(\zeta_{5j}-E[\zeta_{51}])\right\rangle\right] \tag{A-54}$$

$$+\langle E[\zeta_{31}]-\delta_3,E[\zeta_{51}]-\delta_5\rangle,\text{ similarly} \tag{A-55}$$

$$E\left[\left\langle \widehat{\delta}_4(h)-\delta_4,\widehat{\delta}_5(h)-\delta_5\right\rangle\right]=E\left[\left\langle \frac{1}{N}\sum_{i=1}^{N}\varepsilon_{4j}\widehat{f}_{\mathbf{U}}(\mathbf{u}_i)\,\widehat{f}_{V\mathbf{U}}(v_j,\mathbf{u}_j),\frac{1}{N}\sum_{j=1}^{N}\varepsilon_{5j}\widehat{f}^2_{V\mathbf{U}}(v_j,\mathbf{u}_j)\right\rangle\right] \tag{A-56}$$

$$+E\left[\left\langle \frac{1}{N}\sum_{j=1}^{N}(\zeta_{4j}-E[\zeta_{41}]),\frac{1}{N}\sum_{j=1}^{N}(\zeta_{5j}-E[\zeta_{51}])\right\rangle\right] \tag{A-57}$$

$$+\langle E[\zeta_{41}]-\delta_4,E[\zeta_{51}]-\delta_5\rangle \tag{A-58}$$

As it was proven above, terms such as (A-50), (A-51), (A-53), (A-54), (A-56), and (A-57) will not contribute towards the leading terms in the expansion. However, terms (A-52), (A-55) and (A-58) will. In particular, in view of Lemma B-1, it follows

$$\begin{aligned}&\langle E[\zeta_{31}]-\delta_3,E[\zeta_{41}]-\delta_4\rangle\\&=h^{2P}\left\langle\int\pi_3(v,\mathbf{u})\,S_{W\mathcal{K}}(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right.\\&\left.\int\pi_2(\mathbf{u})\,S_{\mathcal{K}}(\mathbf{u})\,f_{\mathbf{U}}(\mathbf{u})\,d\mathbf{u}+\int\pi_3(v,\mathbf{u})\,S_{W\mathcal{K}}(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right\rangle+o\left(N^{-1}h^{P-d}\right),\end{aligned}$$

$$
\begin{aligned}
&\langle E[\zeta_{31}] - \delta_3, E\left[\zeta_{51}\right] - \delta_5\rangle \\
&= 2h^{2P} \left\| \int \pi_3\left(v, \mathbf{u}\right) S_{W\mathcal{K}}\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u} \right\|^2 \\
&+ N^{-1} h^{P-d} \left\langle \int \pi_3\left(v, \mathbf{u}\right) S_{W\mathcal{K}}\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u}, C_{W\mathcal{K}} \int \pi_3\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\langle E[\zeta_{41}] - \delta_4, E\left[\zeta_{51}\right] - \delta_5\rangle \\
&= 2h^{2P} \left\langle \int \pi_2\left(\mathbf{u}\right) S_{\mathcal{K}}\left(\mathbf{u}\right) f_{\mathbf{U}}\left(\mathbf{u}\right) d\mathbf{u} + \int \pi_3\left(v, \mathbf{u}\right) S_{W\mathcal{K}}\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u}, \right. \\
&\left. \int \pi_3\left(v, \mathbf{u}\right) S_{W\mathcal{K}}\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u} \right\rangle \\
&+ N^{-1} h^{P-d} \left\langle \int \pi_2\left(\mathbf{u}\right) S_{\mathcal{K}}\left(\mathbf{u}\right) f_{\mathbf{U}}\left(\mathbf{u}\right) d\mathbf{u} + \int \pi_3\left(v, \mathbf{u}\right) S_{W\mathcal{K}}\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u}, \right. \\
&\left. C_{W\mathcal{K}} \int \pi_3\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u} \right\rangle .
\end{aligned}
$$

Summary

Summarizing, let us define the following quantities:

$$
\begin{aligned}
\mathfrak{B}_{1,1} &= \int \pi_2\left(\mathbf{u}\right) S_{\mathcal{K}}\left(\mathbf{u}\right) f_{\mathbf{U}}\left(\mathbf{u}\right) d\mathbf{u}, \\
\mathfrak{B}_{1,2} &= \int \pi_3\left(v, \mathbf{u}\right) S_{W\mathcal{K}}\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u}, \text{ and} \\
\mathfrak{B}_2 &= C_{W\mathcal{K}} \int \pi_3\left(v, \mathbf{u}\right) f_{V\mathbf{U}}\left(v, \mathbf{u}\right) dvd\mathbf{u},
\end{aligned}
$$

then the contributions of each of the terms analyzed above will be:

Term:	Contribution: h^{2P}	Contribution: $N^{-1}h^{P-d}$	Contribution: $N^{-2}h^{-2d}$
(A-1)	–	–	–
(A-2)	$+4\left\|\mathfrak{B}_{1,1}\right\|^2$	–	–
(A-3)	$+4\left\|\mathfrak{B}_{1,2}\right\|^2$	–	–
(A-4)	$+\left\|\mathfrak{B}_{1,1}+\mathfrak{B}_{1,2}\right\|^2$	–	–
(A-5)	$+4\left\|\mathfrak{B}_{1,2}\right\|^2$	$+4\left\langle\mathfrak{B}_2,\mathfrak{B}_{1,2}\right\rangle$	$\left\|\mathfrak{B}_2\right\|^2$
(A-6)	–	–	–
(A-7)	–	–	–
(A-8)	–	–	–
(A-9)	–	–	–
(A-10)	$-8\left\langle\mathfrak{B}_{1,1},\mathfrak{B}_{1,2}\right\rangle$	–	–
(A-11)	$-4\left\langle\mathfrak{B}_{1,1},\mathfrak{B}_{1,1}+\mathfrak{B}_{1,2}\right\rangle$	–	–
(A-12)	$+8\left\langle\mathfrak{B}_{1,1},\mathfrak{B}_{1,2}\right\rangle$	$+4\left\langle\mathfrak{B}_2,\mathfrak{B}_{1,1}\right\rangle$	–
(A-13)	$+4\left\langle\mathfrak{B}_{1,2},\mathfrak{B}_{1,1}+\mathfrak{B}_{1,2}\right\rangle$	–	–
(A-14)	$-8\left\|\mathfrak{B}_{1,2}\right\|^2$	$-4\left\langle\mathfrak{B}_2,\mathfrak{B}_{1,2}\right\rangle$	–
(A-15)	$-4\left\langle\mathfrak{B}_{1,1}+\mathfrak{B}_{1,2},\mathfrak{B}_{1,2}\right\rangle$	$-2\left\langle\mathfrak{B}_2,\mathfrak{B}_{1,1}+\mathfrak{B}_{1,2}\right\rangle$	–
Net:	$\left\|\mathfrak{B}_{1,1}-\mathfrak{B}_{1,2}\right\|^2$	$2\left\langle\mathfrak{B}_{1,1}-\mathfrak{B}_{1,2},\mathfrak{B}_2\right\rangle$	$\left\|\mathfrak{B}_2\right\|^2$

Therefore, we conclude that the leading terms are:

$$
\begin{aligned}
& h^{2P}\left\|\int\pi_2(\mathbf{u})\,S_{\mathcal{K}}(\mathbf{u})\,f_{\mathbf{U}}(\mathbf{u})\,d\mathbf{u}-\int\pi_3(v,\mathbf{u})\,S_{W\mathcal{K}}(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right\|^2 \\
& +\frac{2C_{W\mathcal{K}}}{Nh^d}h^P\left\langle\int\pi_3(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right. \\
& \left.,\int\pi_2(\mathbf{u})\,S_{\mathcal{K}}(\mathbf{u})\,f_{\mathbf{U}}(\mathbf{u})\,d\mathbf{u}-\int\pi_3(v,\mathbf{u})\,S_{W\mathcal{K}}(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right\rangle \\
& +\frac{C_{W\mathcal{K}}^2}{N^2h^{2d}}\left\|\int\pi_3(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right\|^2 \\
& =h^{2P}\left\|\mathfrak{B}_{1,1}-\mathfrak{B}_{1,2}\right\|^2+2\frac{h^P}{Nh^d}\left\langle\mathfrak{B}_2,\mathfrak{B}_{1,1}-\mathfrak{B}_{1,2}\right\rangle+\frac{1}{N^2h^{2d}}\left\|\mathfrak{B}_2\right\|^2 \\
& =\left\|h^P\mathfrak{B}_1+\mathfrak{B}_2N^{-1}h^{-d}\right\|^2,
\end{aligned}
$$

where $\mathfrak{B}_1=\mathfrak{B}_{1,1}-\mathfrak{B}_{1,2}$, as required.

Appendix B: Technical Lemmas

Lemma B-1 *Let Assumptions (A1)–(A3) hold. Then*

$$E\left(W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right)=f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^P S_{W\mathcal{K}}\left(v,\mathbf{u}\right)+\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right), \tag{B-1}$$

$$E\left(\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right)=f_{\mathbf{U}}\left(\mathbf{u}\right)+h^P S_{\mathcal{K}}\left(\mathbf{u}\right)+\beta_{\mathcal{K}}\left(h,\mathbf{u}\right), \quad \forall\left(v,\mathbf{u}\right)\in\Re^d \tag{B-2}$$

where $\sup_{(v,\mathbf{u})}\left|\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right|=o\left(h^P\right)$*,* $\sup_{\mathbf{U}}\left|\beta_{\mathcal{K}}\left(h,\mathbf{u}\right)\right|=o\left(h^P\right)$ *as* $h\to 0$*, and*

$$q_3=\delta_3+h^P\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}+\gamma_{W\mathcal{K}}\left(h\right), \tag{B-3}$$

$$q_2=\delta_2+h^P\int\pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}+\gamma_{\mathcal{K}}\left(h\right), \tag{B-4}$$

$$q_5=\delta_5+2h^P\left[\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right]+\gamma_{W\mathcal{K}}\left(h\right), \tag{B-5}$$

$$q_4=\delta_4+h^P\left[\int\pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)d\mathbf{u}+\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right]+\gamma_{W\mathcal{K}}\left(h\right), \tag{B-6}$$

where $\left|\gamma_{W\mathcal{K}}\left(h\right)\right|=o\left(h^P\right)$*, and* $\left|\gamma_{\mathcal{K}}\left(h\right)\right|=o\left(h^P\right)$ *as* $h\to 0$*.*

Proof. We prove (B-1) and (B-3) only, as (B-2) and (B-4) follow the exact same arguments. By a simple change of argument,

$$E\left(W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right)=\frac{1}{h^d}\int f_{V\mathbf{U}}\left(v+ch,\mathbf{u}+\mathbf{c}h\right)W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)dcd\mathbf{c}.$$

Assumption (A3) ensures that a Taylor series expansion is valid, and, uniformly in $(c,\mathbf{c})\in[-1,1]^d$,

$$\left|f_{V\mathbf{U}}\left(v+ch,\mathbf{u}+\mathbf{c}h\right)-\sum_{0\leq|\alpha|\leq P}\frac{h^{|\alpha|}}{\alpha!}D^{\alpha}f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\left(c,\mathbf{c}\right)^{\alpha}\right|\leq\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right),$$

where $\sup_{(v,\mathbf{u})}\left|\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right|=o\left(h^P\right)$, with $h\to 0$. We use the notation $\alpha=(\alpha_1,\ldots,\alpha_d)$, $\alpha!=\alpha_1!\times\ldots\times\alpha_d!$, $|\alpha|=\sum_{j=1}^{d}\alpha_j$, $(c,\mathbf{c})^{\alpha}=c^{\alpha_1}\times c_1^{\alpha_2}\times\ldots\times c_{d-1}^{\alpha_d}$, $\sum_{0\leq|\alpha|\leq P}=\sum_{j=0}^{P}\sum_{\substack{\alpha_1=0\\ \alpha_1+\ldots+\alpha_d=j}}^{j}\cdots\sum_{\alpha_d=0}^{j}$, and

$$D^{\alpha}f_{V\mathbf{U}}\left(v,\mathbf{u}\right)=\frac{\partial^{\alpha}f_{V\mathbf{U}}\left(v,\mathbf{u}\right)}{\partial v^{\alpha_1}\partial u_1^{\alpha_2}\ldots\partial u_{d-1}^{\alpha_d}}.$$

It follows from Assumption (A2),

$$\begin{aligned}E\left(W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right)&=f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+\sum_{|\alpha|=P}\frac{h^{|\alpha|}}{\alpha!}D^{\alpha}f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\\&\quad\times\int\left(c,\mathbf{c}\right)^{\alpha}W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)dcd\mathbf{c}+\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\\&=f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^P S_{W\mathcal{K}}\left(v,\mathbf{u}\right)+\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right).\end{aligned}$$

Given this expression, It also follows

$$
\begin{aligned}
q_3 &= E\left[\pi_3\left(V_1,\mathbf{U}_1\right)W_h\left(V_2-V_1\right)\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{U}_1\right)\right] \\
&= E\left[\pi_3\left(V_1,\mathbf{U}_1\right)E\left[W_h\left(V_2-V_1\right)\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{U}_1\right)\right|V=V_1,\mathbf{U}=\mathbf{U}_1\right]\right] \\
&= E\left[\pi_3\left(V_1,\mathbf{U}_1\right)\left(f_{V\mathbf{U}}\left(V_1,\mathbf{U}_1\right)+h^P S_{W\mathcal{K}}\left(V_1,\mathbf{U}_1\right)+\beta_{W\mathcal{K}}\left(h,\left(V_1,\mathbf{U}_1\right)\right)\right)\right] \\
&= \delta_3 + h^P\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}+\gamma_{W\mathcal{K}}\left(h\right),
\end{aligned}
$$

where the last equality follows from $\sup_{(v,\mathbf{u})}\left|\beta_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right)\right| = o\left(h^P\right)$, and $\int\pi_3\left(v,\mathbf{u}\right)\times f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}<\infty$ since π_3 and $f_{V\mathbf{U}}$ are bounded on the compact support $\Omega_{V\mathbf{U}}$. Similarly, we have

$$
\begin{aligned}
q_4 &= E\left[\pi_4\left(V_1,\mathbf{U}_1\right)W_h\left(V_2-V_1\right)\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\right] \\
&= E\left[\pi_4\left(V_1,\mathbf{U}_1\right)E\left[W_h\left(V_2-V_1\right)\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{U}_1\right)\right|V=V_1,\mathbf{U}=\mathbf{U}_1\right] \\
&\times E\left[\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\right|V=V_1,\mathbf{U}=\mathbf{U}_1]] \\
&= E\left[\pi_4\left(V_1,\mathbf{U}_1\right)\left(f_{V\mathbf{U}}\left(V_1,\mathbf{U}_1\right)+h^P S_{W\mathcal{K}}\left(V_1,\mathbf{U}_1\right)+\beta_{W\mathcal{K}}\left(h,\left(V_1,\mathbf{U}_1\right)\right)\right)\right. \\
&\times\left.\left(f_{\mathbf{U}}\left(\mathbf{U}_1\right)+h^P S_{\mathcal{K}}\left(\mathbf{U}_1\right)+\beta_{\mathcal{K}}\left(h,\mathbf{U}_1\right)\right)\right] \\
&= \int\pi_4\left(v,\mathbf{u}\right)f_{V\mathbf{U}}^2\left(v,\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)dvd\mathbf{u}+h^P\int\pi_4\left(v,\mathbf{u}\right)f_{V\mathbf{U}}^2\left(v,\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)dvd\mathbf{u} \\
&+ h^P\int\pi_4\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)dvd\mathbf{u}+\gamma_{W\mathcal{K}}\left(h\right) \\
&= \delta_4 + h^P\left[\int\pi_2\left(\mathbf{u}\right)S_{\mathcal{K}}\left(\mathbf{u}\right)d\mathbf{u}+\int\pi_3\left(v,\mathbf{u}\right)S_{W\mathcal{K}}\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)dvd\mathbf{u}\right]+\gamma_{W\mathcal{K}}\left(h\right),
\end{aligned}
$$

where the last equality follows from observing that

$$
\begin{aligned}
\pi_4\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right) &= E\left[\left.\frac{\omega}{f_{V\mathbf{U}}^2\left(V,\mathbf{U}\right)}\right|V=v,\mathbf{U}=\mathbf{u}\right]f_{V\mathbf{U}}\left(v,\mathbf{u}\right) \\
&= E\left[\left.\frac{\omega}{f_{V\mathbf{U}}\left(V,\mathbf{U}\right)}\right|V=v,\mathbf{U}=\mathbf{u}\right] \\
&= \widetilde{\pi}_2\left(v,\mathbf{u}\right), \text{ such that} \\
E\left[\widetilde{\pi}_2\left(V,\mathbf{U}\right)\right|\mathbf{U}=\mathbf{u}] &= \int\widetilde{\pi}_2\left(v,\mathbf{u}\right)f_{V|\mathbf{U}}\left(v|\,\mathbf{u}\right)dv=\pi_2\left(\mathbf{u}\right), \text{ and} \\
\pi_4\left(v,\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right) &= E\left[\left.\frac{\omega}{f_{V\mathbf{U}}^2\left(V,\mathbf{U}\right)}\right|V=v,\mathbf{U}=\mathbf{u}\right]f_{\mathbf{U}}\left(\mathbf{u}\right) \\
&= E\left[\left.\frac{\omega f_{\mathbf{U}}\left(\mathbf{U}\right)}{f_{V\mathbf{U}}^2\left(V,\mathbf{U}\right)}\right|V=v,\mathbf{U}=\mathbf{u}\right] \\
&= \pi_3\left(v,\mathbf{u}\right).
\end{aligned}
$$

By the exact same arguments, we have

$$
\begin{aligned}
q_5 &= E\left[\pi_5(V_1,\mathbf{U}_1)\left\|E\left[W_h(V_2-V_1)\mathcal{K}_h(\mathbf{U}_2-\mathbf{U}_1)\right|V=V_1,\mathbf{U}=\mathbf{U}_1\right]\right\|^2\Big] \\
&= E\left[\pi_5(V_1,\mathbf{U}_1)\left(f_{V\mathbf{U}}(V_1,\mathbf{U}_1)+h^P S_{W\mathcal{K}}(V_1,\mathbf{U}_1)+\beta_{W\mathcal{K}}(h,(V_1,\mathbf{U}_1))\right)^2\right] \\
&= \int \pi_5(v,\mathbf{u})\, f_{V\mathbf{U}}^3(v,\mathbf{u})\,dvd\mathbf{u}+2h^P\left[\int \pi_5(v,\mathbf{u})\,S_{W\mathcal{K}}(V_1,\mathbf{U}_1)\,f_{V\mathbf{U}}^2(v,\mathbf{u})\,dvd\mathbf{u}\right]+\gamma_{W\mathcal{K}}(h) \\
&= \delta_5+2h^P\left[\int \pi_3(v,\mathbf{u})\,S_{W\mathcal{K}}(v,\mathbf{u})\,f_{V\mathbf{U}}(v,\mathbf{u})\,dvd\mathbf{u}\right]+\gamma_{W\mathcal{K}}(h).
\end{aligned}
$$

as needed. ■

Lemma B-2 *Let Assumptions (A1)–(A3) hold. Then*

$$
\begin{aligned}
E\left[W_h(V_1-v)\mathcal{K}_h^2(\mathbf{U}_1-\mathbf{u})\right] &= C_{\mathcal{K}} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-(d-1)}+\psi^*_{W\mathcal{K},12}(h,(v,\mathbf{u})), \\
E\left[W_h(V_1-v)\mathcal{K}_h^3(\mathbf{U}_1-\mathbf{u})\right] &= C_{\mathcal{K},3} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-2(d-1)}+\psi^*_{W\mathcal{K},13}(h,(v,\mathbf{u})), \\
E\left[W_h^2(V_1-v)\mathcal{K}_h^2(\mathbf{U}_1-\mathbf{u})\right] &= C_{W\mathcal{K}} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-d}+\psi_{W\mathcal{K}}(h,(v,\mathbf{u})), \\
E\left[W_h^2(V_1-v)\mathcal{K}_h^3(\mathbf{U}_1-\mathbf{u})\right] &= C_{W\mathcal{K},23} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-(2d-1)}+\psi_{W\mathcal{K},23}(h,(v,\mathbf{u})), \\
E\left[W_h^3(V_1-v)\mathcal{K}_h^3(\mathbf{U}_1-\mathbf{u})\right] &= C_{W\mathcal{K},33} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-2d}+\psi_{W\mathcal{K},33}(h,(v,\mathbf{u})), \\
E\left[W_h^3(V_1-v)\mathcal{K}_h^4(\mathbf{U}_1-\mathbf{u})\right] &= C_{W\mathcal{K},34} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-(3d-1)}+\psi_{W\mathcal{K},34}(h,(v,\mathbf{u})), \\
E\left[W_h^4(V_1-v)\mathcal{K}_h^4(\mathbf{U}_1-\mathbf{u})\right] &= C_{W\mathcal{K},44} f_{V\mathbf{U}}(v,\mathbf{u})\,h^{-3d}+\psi_{W\mathcal{K},44}(h,(v,\mathbf{u})),
\end{aligned}
$$

where $\sup_{(v,\mathbf{u})}\left|\psi^*_{W\mathcal{K},12}(h,(v,\mathbf{u}))\right| = o\left(h^{-(d-1)}\right)$, $\sup_{(v,\mathbf{u})}\left|\psi^*_{W\mathcal{K},13}(h,(v,\mathbf{u}))\right| = o\left(h^{-2(d-1)}\right)$, $\sup_{(v,\mathbf{u})}\left|\psi_{W\mathcal{K}}(h,(v,\mathbf{u}))\right| = o\left(h^{-d}\right)$, $\sup_{(v,\mathbf{u})}\left|\psi_{W\mathcal{K},23}(h,(v,\mathbf{u}))\right| = o\left(h^{-(2d-1)}\right)$, $\sup_{(v,\mathbf{u})}\left|\psi_{W\mathcal{K},33}(h,(v,\mathbf{u}))\right| = o\left(h^{-2d}\right)$, $\sup_{(v,\mathbf{u})}\left|\psi_{W\mathcal{K},34}(h,(v,\mathbf{u}))\right| = o\left(h^{-(3d-1)}\right)$, *and* $\sup_{(v,\mathbf{u})}\left|\psi_{W\mathcal{K},44}(h,(v,\mathbf{u}))\right| = o\left(h^{-3d}\right)$.

Proof. Firstly,

$$
\begin{aligned}
E\left[W_h^2(V_1-v)\mathcal{K}_h^2(\mathbf{U}_1-\mathbf{u})\right] &= \frac{1}{h^{2d}}\int W^2\left(\frac{t-v}{h}\right)\mathcal{K}^2\left(\frac{\mathbf{t}-\mathbf{u}}{h}\right) f_{V\mathbf{U}}(t,\mathbf{t})\,dtd\mathbf{t} \\
&= \frac{1}{h^d}\int W^2(c)\,\mathcal{K}^2(\mathbf{c})\,f_{V\mathbf{U}}(v+ch,\mathbf{u}+\mathbf{c}h)\,dcd\mathbf{c}.
\end{aligned}
$$

Similarly, we have that

$$
\begin{aligned}
E\left[W_h(V_1-v)\mathcal{K}_h^2(\mathbf{U}_1-\mathbf{u})\right] &= \frac{1}{h^{2d-1}}\int W\left(\frac{t-v}{h}\right)\mathcal{K}^2\left(\frac{\mathbf{t}-\mathbf{u}}{h}\right) f_{V\mathbf{U}}(t,\mathbf{t})\,dtd\mathbf{t} \\
&= \frac{1}{h^{d-1}}\int W(c)\,\mathcal{K}^2(\mathbf{c})\,f_{V\mathbf{U}}(v+ch,\mathbf{u}+\mathbf{c}h)\,dcd\mathbf{c}.
\end{aligned}
$$

It follows from Assumption (A3), that $f_{V\mathbf{U}}$, and $f_{\mathbf{U}}$ are Lipschitz continuous on $\Omega_{V\mathbf{U}}$ and $\Omega_{\mathbf{U}}$ respectively, with some Lipschitz constants $L_{f_{V\mathbf{U}}}$ and $L_{f_{\mathbf{U}}}$. Thus,

$$\begin{aligned}
&\left|\frac{1}{h^d}\int W^2(c)\,\mathcal{K}^2(\mathbf{c})\,f_{V\mathbf{U}}(v+ch,\mathbf{u}+\mathbf{c}h)\,dcd\mathbf{c}-\frac{1}{h^d}C_{W\mathcal{K}}f_{V\mathbf{U}}(v,\mathbf{u})\right|\\
&\leq \frac{L_{f_{V\mathbf{U}}}}{h^{d-1}}\int \left\|W(c)\,\mathcal{K}(\mathbf{c})\right\|^2\left\|(c,\mathbf{c})\right\|dcd\mathbf{c},\\
&\left|\frac{1}{h^{d-1}}\int W(c)\,\mathcal{K}^2(\mathbf{c})\,f_{V\mathbf{U}}(v+ch,\mathbf{u}+\mathbf{c}h)\,dcd\mathbf{c}-\frac{1}{h^{d-1}}C_{\mathcal{K}}f_{V\mathbf{U}}(v,\mathbf{u})\right|\\
&\leq \frac{L_{f_{V\mathbf{U}}}}{h^{d-2}}\int \left\|W(c)\right\|\left\|\mathcal{K}(\mathbf{c})\right\|^2\left\|(c,\mathbf{c})\right\|dcd\mathbf{c},
\end{aligned}$$

which proves this Lemma. The same argument applies to each of the other components. ■

Lemma B-3 *Let Assumptions (A1)–(A3) hold. Then*

$$\begin{aligned}
&E\left[W_h(V-v_1)\,\mathcal{K}_h(\mathbf{U}-\mathbf{u}_1)\,W_h(V-v_2)\,\mathcal{K}_h(\mathbf{U}-\mathbf{u}_2)\right] \qquad \text{(B-7)}\\
&=h^{-d}\left\langle W\mathcal{K},W\mathcal{K}\right\rangle\left(\frac{v_1-v_2}{h},\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}(v_1,\mathbf{u}_1)+\psi_{\langle W\mathcal{K},W\mathcal{K}\rangle}(h,(v,\mathbf{u})),\\
&E\left[\mathcal{K}_h(\mathbf{U}-\mathbf{u}_1)\,\mathcal{K}_h(\mathbf{U}-\mathbf{u}_2)\right]\\
&=h^{-(d-1)}\left\langle \mathcal{K},\mathcal{K}\right\rangle\left(\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{\mathbf{U}}(\mathbf{u}_1)+\psi_{\langle \mathcal{K},\mathcal{K}\rangle}(h,\mathbf{u})
\end{aligned}$$

$$\begin{aligned}
&E\left[\mathcal{K}_h(\mathbf{U}-\mathbf{u}_1)\,W_h(V-v_2)\,\mathcal{K}_h(\mathbf{U}-\mathbf{u}_2)\right]\\
&=h^{-(d-1)}\left\langle \mathcal{K},W\mathcal{K}\right\rangle\left(\frac{v_1-v_2}{h},\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}(v_1,\mathbf{u}_1)+\psi_{\langle \mathcal{K},W\mathcal{K}\rangle}(h,(v,\mathbf{u})),\\
&E\left[\mathcal{K}_h(\mathbf{U}-\mathbf{u}_1)\,W_h^2(V-v_2)\,\mathcal{K}_h^2(\mathbf{U}-\mathbf{u}_2)\right]\\
&=h^{-(2d-1)}\left\langle \mathcal{K},W^2\mathcal{K}^2\right\rangle\left(\frac{v_1-v_2}{h},\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}(v_1,\mathbf{u}_1)+\psi_{\langle \mathcal{K},W^2\mathcal{K}^2\rangle}(h,(v,\mathbf{u})),\\
&E\left[W_h(V-v_1)\,\mathcal{K}_h(\mathbf{U}-\mathbf{u}_1)\,W_h^2(V-v_2)\,\mathcal{K}_h^2(\mathbf{U}-\mathbf{u}_2)\right]\\
&=h^{-2d}\left\langle W\mathcal{K},W^2\mathcal{K}^2\right\rangle\left(\frac{v_1-v_2}{h},\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}(v_1,\mathbf{u}_1)+\psi_{\langle W\mathcal{K},W^2\mathcal{K}^2\rangle}(h,(v,\mathbf{u})),
\end{aligned}$$

where $\langle f,g\rangle(v,u)=\int f(c,\mathbf{c})\,g(c-v,\mathbf{c}-\mathbf{u})\,dcd\mathbf{c}$, *and*

$$\begin{aligned}
\sup_{\Omega_{V\mathbf{U}}}\psi_{\langle W\mathcal{K},W\mathcal{K}\rangle}(h,(v,\mathbf{u}))&=o\left(h^{-d}\right),\\
\sup_{\Omega_{\mathbf{U}}}\psi_{\langle \mathcal{K},\mathcal{K}\rangle}(h,\mathbf{u})&=o\left(h^{-(d-1)}\right),\\
\sup_{\Omega_{V\mathbf{U}}}\psi_{\langle \mathcal{K},W\mathcal{K}\rangle}(h,(v,\mathbf{u}))&=o\left(h^{-(d-1)}\right),\\
\sup_{\Omega_{V\mathbf{U}}}\psi_{\langle \mathcal{K},W^2\mathcal{K}^2\rangle}(h,(v,\mathbf{u}))&=o\left(h^{-(2d-1)}\right),\\
\sup_{\Omega_{V\mathbf{U}}}\psi_{\langle W\mathcal{K},W^2\mathcal{K}^2\rangle}(h,(v,\mathbf{u}))&=o\left(h^{-2d}\right),
\end{aligned}$$

Proof. We only show the working for (B-7), as all other terms follow the exact same argument. Firstly,

$$\begin{aligned}
&E\left[W_h\left(V-v_1\right)\mathcal{K}_h\left(\mathbf{U}-\mathbf{u}_1\right)W_h\left(V-v_2\right)\mathcal{K}_h\left(\mathbf{U}-\mathbf{u}_2\right)\right]\\
&=\frac{1}{h^{2d}}\int W\left(\frac{t-v_1}{h}\right)\mathcal{K}\left(\frac{\mathbf{t}-\mathbf{u}_1}{h}\right)W\left(\frac{t-v_2}{h}\right)\mathcal{K}\left(\frac{\mathbf{t}-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}\left(t,\mathbf{t}\right)dtd\mathbf{t}\\
&=\frac{1}{h^{d}}\int W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)W\left(c+\frac{v_1-v_2}{h}\right)\mathcal{K}\left(\mathbf{c}+\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}\left(v_1+ch,\mathbf{u}_1+\mathbf{c}h\right)dcd\mathbf{c},
\end{aligned}$$

It follows from Assumption (A3), that $f_{V\mathbf{U}}$, and $f_{\mathbf{U}}$ are Lipschitz continuous on $\Omega_{V\mathbf{U}}$ and $\Omega_{\mathbf{U}}$ respectively, with some Lipschitz constants $L_{f_{V\mathbf{U}}}$ and $L_{f_{\mathbf{U}}}$. Similarly, from Assumption (A2), $W\mathcal{K}$ is also Lipschitz continuous on $[0,1]^d$, with some Lipschitz constant $L_{W\mathcal{K}}$. Thus,

$$\begin{aligned}
&\left|\frac{1}{h^{d}}\int W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)W\left(c+\frac{v_1-v_2}{h}\right)\mathcal{K}\left(\mathbf{c}+\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}\left(v_1+ch,\mathbf{u}_1+\mathbf{c}h\right)dcd\mathbf{c}\right.\\
&\left.-\frac{1}{h^d}\left\langle W\mathcal{K},W\mathcal{K}\right\rangle\left(\frac{v_1-v_2}{h},\frac{\mathbf{u}_1-\mathbf{u}_2}{h}\right)f_{V\mathbf{U}}\left(v_1,\mathbf{u}_1\right)\right|\\
&\leq\frac{L_{W\mathcal{K}}L_{f_{V\mathbf{U}}}}{h^{d-1}}\int\left\|W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)\right\|\left\|\left(c,\mathbf{c}\right)\right\|dcd\mathbf{c}.
\end{aligned}$$

■

Lemma B-4 *Let Assumptions (A1)–(A3) hold. Then*

$$\begin{aligned}
E\left[\left\|\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right\|^2\right]&=C_{\mathcal{K}}f_{\mathbf{U}}\left(\mathbf{u}\right)h^{-(d-1)}+\psi_{\mathcal{K}}\left(h,\mathbf{u}\right),\\
E\left[\mathcal{K}_h^3\left(\mathbf{U}_1-\mathbf{u}\right)\right]&=C_{\mathcal{K},3}f_{\mathbf{U}}\left(\mathbf{u}\right)h^{-2(d-1)}+\psi_{\mathcal{K},3}\left(h,\mathbf{u}\right),\\
E\left[\left\|\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right\|^4\right]&=C_{\mathcal{K},4}f_{\mathbf{U}}\left(\mathbf{u}\right)h^{-3(d-1)}+\psi_{\mathcal{K},4}\left(h,\mathbf{u}\right),
\end{aligned}$$

where $\sup_{\mathbf{u}}\left|\psi_{\mathcal{K}}\left(h,\mathbf{u}\right)\right|=o\left(h^{-(d-1)}\right)$, $\sup_{\mathbf{u}}\left|\psi_{\mathcal{K},3}\left(h,\mathbf{u}\right)\right|=o\left(h^{-2(d-1)}\right)$, *and* $\sup_{\mathbf{u}}\left|\psi_{\mathcal{K},4}\left(h,\mathbf{u}\right)\right|=o\left(h^{-3(d-1)}\right)$.

Proof. These are special cases of those in Lemma B-2, with $W_h(\cdot)$ empty. The result follows by the same arguments. ■

Lemma B-5 *Let Assumptions (A1)–(A3) hold. Then*

$$E\left[\pi_3\left(V_1,\mathbf{U}_1\right)W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]=\pi_3\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^P\widetilde{S}_{W\mathcal{K},3}\left(v,\mathbf{u}\right)+\widetilde{\beta}_{W\mathcal{K},3}\left(h,\left(v,\mathbf{u}\right)\right), \tag{B-8}$$

$$E\left[\pi_5\left(V_1,\mathbf{U}_1\right)W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]=\pi_5\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^P\widetilde{S}_{W\mathcal{K},5}\left(v,\mathbf{u}\right)+\widetilde{\beta}_{W\mathcal{K},5}\left(h,\left(v,\mathbf{u}\right)\right),$$

$$E\left[\pi_2\left(\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]=\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)+h^P\widetilde{S}_{\mathcal{K},2}\left(\mathbf{u}\right)+\widetilde{\beta}_{\mathcal{K},2}\left(h,\mathbf{u}\right), \tag{B-9}$$

$$E\left[\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]=\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)+h^P\widetilde{S}^*_{\mathcal{K},2}\left(\mathbf{u}\right)+\widetilde{\beta}^*_{\mathcal{K},2}\left(h,\mathbf{u}\right), \tag{B-10}$$

$$E\left[\pi_5\left(V_2,\mathbf{U}_2\right)W_h^2\left(V_2-v\right)\mathcal{K}_h^2\left(\mathbf{U}_2-\mathbf{u}\right)\right]=h^{-d}C_{W\mathcal{K}}\pi_5\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+\beta_{W\mathcal{K},5}\left(h,\left(v,\mathbf{u}\right)\right) \tag{B-11}$$

where

$$\widetilde{S}_{W\mathcal{K},l}\left(v,\mathbf{u}\right)=\frac{1}{P!}\left[d_W\frac{\partial^P}{\partial v^P}\left[\pi_l\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\right]+d_K\sum_{j=1}^{d-1}\frac{\partial^P}{\partial u_j^P}\left[\pi_l\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\right]\right],$$

$$\widetilde{S}_{\mathcal{K},2}\left(\mathbf{u}\right)=\frac{d_K}{P!}\sum_{j=1}^{d-1}\frac{\partial^P}{\partial u_j^P}\left[\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)\right],$$

$$\widetilde{S}^*_{\mathcal{K},2}\left(\mathbf{u}\right)=\frac{d_K}{P!}\sum_{j=1}^{d-1}\int\frac{\partial^P}{\partial u_j^P}\left[\widetilde{\pi}_2\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\right]dv,$$

and $\sup_{(v,\mathbf{u})}\left|\widetilde{\beta}_{W\mathcal{K},l}\left(h,\left(v,\mathbf{u}\right)\right)\right|=o\left(h^{-d}\right)$*, for* $l=3$ *and* 5*,* $\sup_{\mathbf{U}}\left|\widetilde{\beta}_{\mathcal{K}}\left(h,\mathbf{u}\right)\right|=o\left(h^{-(d-1)}\right)$*,* $\sup_{(v,\mathbf{u})}\left|\widetilde{\beta}^*_{\mathcal{K}}\left(h,\mathbf{u}\right)\right|=o\left(h^{-(d-1)}\right)$*, and* $\sup_{(v,\mathbf{u})}\left|\beta_{W\mathcal{K},5}\left(h,\left(v,\mathbf{u}\right)\right)\right|=o\left(h^{-d}\right)$ *as* $h\to 0$*.*

Proof. As before, we only show the results for (B-8), (B-10), and (B-11) as the others follow the exact same arguments. By a simple change of argument

$$\begin{aligned}&E\left[\pi_3\left(V_1,\mathbf{U}_1\right)W_h\left(V_1-v\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]\\&=\int\pi_3\left(v+ch,\mathbf{u}+\mathbf{c}h\right)f_{V\mathbf{U}}\left(v+ch,\mathbf{u}+\mathbf{c}h\right)W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)dcd\mathbf{c}\\&=\pi_3\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+h^P\sum_{|\alpha|=P}\frac{1}{\alpha!}D^\alpha\left[\pi_3\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\right]\\&\times\int\left(c,\mathbf{c}\right)^\alpha W\left(c\right)\mathcal{K}\left(\mathbf{c}\right)dcd\mathbf{c}+\widetilde{\beta}_{W\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right),\end{aligned}$$

where the last equality follows from Assumptions (A3) and (A2). Similarly, we can write (B-10) as

$$\begin{aligned}E\left[\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{u}\right)\right]&=\int\widetilde{\pi}_2\left(v,\mathbf{u}+\mathbf{c}h\right)f_{V\mathbf{U}}\left(v,\mathbf{u}+\mathbf{c}h\right)\mathcal{K}\left(\mathbf{c}\right)dvd\mathbf{c}\\&=\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)+h^P\sum_{|\alpha|=P}\frac{1}{\alpha!}\int D^\alpha_{\mathbf{U}}\left[\widetilde{\pi}_2\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\right]dv\\&\times\int\mathbf{c}^\alpha\mathcal{K}\left(\mathbf{c}\right)d\mathbf{c}+\widetilde{\beta}^*_{\mathcal{K}}\left(h,\left(v,\mathbf{u}\right)\right).\end{aligned}$$

Also,

$$
\begin{aligned}
&E\left[\pi_5\left(V_2,\mathbf{U}_2\right)W_h^2\left(V_2-v\right)\mathcal{K}_h^2\left(\mathbf{U}_2-\mathbf{u}\right)\right]\\
&=h^{-d}\int\pi_5\left(v+ch,\mathbf{u}+\mathbf{c}h\right)f_{V\mathbf{U}}\left(v+ch,\mathbf{u}+\mathbf{c}h\right)W^2\left(c\right)\left\|\mathcal{K}\left(\mathbf{c}\right)\right\|^2dcd\mathbf{c}\\
&=h^{-d}C_{W\mathcal{K}}\pi_5\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)+\widetilde{\beta}_{W\mathcal{K},5}\left(h,\left(v,\mathbf{u}\right)\right)
\end{aligned}
$$

as required. ■

Lemma B-6 *Let Assumptions (A1)–(A3) hold. Then as* $h\to 0$*,*

$$
\begin{aligned}
\mathcal{B}_{3,I}&=\int\left\|\pi_3\left(v,\mathbf{u}\right)\right\|^2f_{V\mathbf{U}}^3\left(v,\mathbf{u}\right)d\mathbf{u}+O\left(h^P\right),\\
\mathcal{B}_{3,III}&=\int\left\|\pi_3\left(v,\mathbf{u}\right)\right\|^2f_{V\mathbf{U}}^3\left(v,\mathbf{u}\right)d\mathbf{u}+O\left(h^P\right),\\
\mathcal{B}_{3,IV}&=\frac{1}{h^d}\left[C_{W\mathcal{K}}\int\left\|\pi_3\left(v,\mathbf{u}\right)f_{V\mathbf{U}}\left(v,\mathbf{u}\right)\right\|^2d\mathbf{u}\left\{1+o\left(1\right)\right\}\right],\\
\mathcal{B}_{2,I}&=\int\left\|\pi_2\left(\mathbf{u}\right)\right\|^2f_{\mathbf{U}}^3\left(\mathbf{u}\right)d\mathbf{u}+O\left(h^P\right),\\
\mathcal{B}_{2,III}&=\int\left\|\pi_2\left(\mathbf{u}\right)\right\|^2f_{\mathbf{U}}^3\left(\mathbf{u}\right)d\mathbf{u}+O\left(h^P\right),\\
\mathcal{B}_{2,IV}&=\frac{1}{h^{d-1}}\left[C_{\mathcal{K}}\int\left\|\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)\right\|^2d\mathbf{u}\left\{1+o\left(1\right)\right\}\right].
\end{aligned}
$$

Proof. For this Lemma, we show the results for $\mathcal{B}_{2,l}$, for $l=I,III,IV$, for notational convenience. The proof of $\mathcal{B}_{3,l}$, for $l=I,III,IV$ follows the exact same arguments, and therefore is omitted.

By Lemma B-5, it follows

$$
\begin{aligned}
\mathcal{B}_{2,I}&=E\left[\left\|E\left[\pi_2\left(\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\middle|\,\mathbf{U}_3\right]\right\|^2\right]\\
&=\int f_{\mathbf{U}}\left(\mathbf{u}\right)\left\|E\left[\pi_2\left(\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{u}-\mathbf{U}_1\right)\right]\right\|^2d\mathbf{u}\\
&=\int f_{\mathbf{U}}\left(\mathbf{u}\right)\left\|\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)+h^P\widetilde{S}_{\mathcal{K}}\left(\mathbf{u}\right)+\widetilde{\beta}_{\mathcal{K}}\left(h,\mathbf{u}\right)\right\|^2d\mathbf{u}\\
&=\int\left\|\pi_2\left(\mathbf{u}\right)f_{\mathbf{U}}\left(\mathbf{u}\right)\right\|^2f_{\mathbf{U}}\left(\mathbf{u}\right)d\mathbf{u}+O\left(h^P\right).
\end{aligned}
$$

Using Lemmas B-1 and B-5, we obtain

$$\begin{aligned}
\mathcal{B}_{2,III} &= E\left[\langle \pi_2(\mathbf{U}_1), \pi_2(\mathbf{U}_2)\rangle \mathcal{K}_h(\mathbf{U}_2-\mathbf{U}_1)\mathcal{K}_h(\mathbf{U}_3-\mathbf{U}_2)\right] \\
&= \int \langle \pi_2(\mathbf{x}), \pi_2(\mathbf{y})\rangle \mathcal{K}_h(\mathbf{y}-\mathbf{x})\mathcal{K}_h(\mathbf{z}-\mathbf{y}) f_{\mathbf{U}}(\mathbf{x}) f_{\mathbf{U}}(\mathbf{y}) f_{\mathbf{U}}(\mathbf{z})\, d\mathbf{x}d\mathbf{y}d\mathbf{z} \\
&= \int \left\langle \int \pi_2(\mathbf{x})\mathcal{K}_h(\mathbf{y}-\mathbf{x}) f_{\mathbf{U}}(\mathbf{x})\, d\mathbf{x}, \int \pi_2(\mathbf{y})\mathcal{K}_h(\mathbf{z}-\mathbf{y}) f_{\mathbf{U}}(\mathbf{z})\, d\mathbf{z}\right\rangle f_{\mathbf{U}}(\mathbf{y})\, d\mathbf{y} \\
&= \int \Big\langle \pi_2(\mathbf{y}) f_{\mathbf{U}}(\mathbf{y}) + h^P \widetilde{S}_{\mathcal{K}}(\mathbf{y}) + \widetilde{\beta}_{\mathcal{K}}(h,\mathbf{y}), \\
&\quad \pi_2(\mathbf{y})\left[f_{\mathbf{U}}(\mathbf{y}) + h^P S_{\mathcal{K}}(\mathbf{y}) + \beta_{\mathcal{K}}(h,\mathbf{y})\right]\Big\rangle f_{\mathbf{U}}(\mathbf{y})\, d\mathbf{y} \\
&= \int \left\|\pi_2(\mathbf{u}) f_{\mathbf{U}}(\mathbf{u})\right\|^2 f_{\mathbf{U}}(\mathbf{u})\, d\mathbf{u} + O\left(h^P\right).
\end{aligned}$$

Finally, by the change of variables: $\mathbf{c} = (\mathbf{x}-\mathbf{y})\, h^{-1}$, we have

$$\begin{aligned}
\mathcal{B}_{2,IV} &= E\left[\langle \pi_2(\mathbf{U}_1), \pi_2(\mathbf{U}_2)\rangle \left\|\mathcal{K}_h(\mathbf{U}_1-\mathbf{U}_2)\right\|^2\right] \\
&= \int \langle \pi_2(\mathbf{x}), \pi_2(\mathbf{y})\rangle \left\|\mathcal{K}_h(\mathbf{x}-\mathbf{y})\right\|^2 f_{\mathbf{U}}(\mathbf{x}) f_{\mathbf{U}}(\mathbf{y})\, d\mathbf{x}d\mathbf{y} \\
&= \frac{1}{h^{d-1}} \int \langle \pi_2(\mathbf{y}+\mathbf{c}h), \pi_2(\mathbf{y})\rangle f_{\mathbf{U}}(\mathbf{y}+\mathbf{c}h) f_{\mathbf{U}}(\mathbf{y}) \left\|\mathcal{K}(\mathbf{c})\right\|^2 d\mathbf{y}d\mathbf{c}.
\end{aligned}$$

From Assumption (A3), it follows that $f_{\mathbf{U}}$ and π_2 are Lipschitz continuous on $\Omega_{\mathbf{U}}$. Then

$$\begin{aligned}
\mathcal{B}_{2,IV} &= \frac{1}{h^{d-1}}\left[\int \left\|\pi_2(\mathbf{u}) f_{\mathbf{U}}(\mathbf{u})\right\|^2 d\mathbf{u} \int \left\|\mathcal{K}(\mathbf{c})\right\|^2 d\mathbf{c}\left\{1+O(h)\right\}\right] \\
&= \frac{1}{h^{d-1}}\left[\int \left\|\pi_2(\mathbf{u}) f_{\mathbf{U}}(\mathbf{u})\right\|^2 d\mathbf{u} \int \left\|\mathcal{K}(\mathbf{c})\right\|^2 d\mathbf{c}\left\{1+o(1)\right\}\right],
\end{aligned}$$

which concludes the proof. ■

Lemma B-7 *Let Assumptions (A1)–(A3) hold. Then as $h \to 0$,*

$$\begin{aligned}
\mathcal{B}_{5,I} &= O\left(h^{-3d}\right), \\
\mathcal{B}_{5,II} &= O\left(Nh^{-2d}\right), \\
\mathcal{B}_{5,III} &= O\left(Nh^{-2d}\right), \\
\mathcal{B}_{5,IV} &= O\left(N^2h^{-d}\right), \\
\mathcal{B}_{5,V} &= O\left(Nh^{-2d}\right) + O\left(N^2h^{-d}\right) + O\left(N^3\right) + O\left(N^3h^P\right), \\
\mathcal{B}_{5,VI} &= O\left(Nh^{-2d}\right) + O\left(N^2h^{-d}\right).
\end{aligned}$$

Proof. As before, it follows from Lemma B-2,

$$
\begin{aligned}
& E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{4} \mathcal{K}_{h;12}^{4}\right] \\
& = E\left[\pi_5\left(V_1, \mathbf{U}_1\right) E\left[\pi_5\left(V_2, \mathbf{U}_2\right) W_h^4\left(V_1 - V_2\right) \mathcal{K}_h^4\left(\mathbf{U}_1 - \mathbf{U}_2\right) \middle| \mathbf{U}_1\right]\right] \\
& = E\left[\pi_5\left(V_1, \mathbf{U}_1\right)\left(h^{-3d} C_{W\mathcal{K},44} \pi_5\left(V_1, \mathbf{U}_1\right) f_{V\mathbf{U}}\left(V_1, \mathbf{U}_1\right) + \widetilde{\beta}_{W\mathcal{K},44}\left(h, \left(V_1, \mathbf{U}_1\right)\right)\right)\right] \\
& = \frac{C_{W\mathcal{K}}}{h^{-3d}} \int \left\|\pi_5\left(v, \mathbf{u}\right)\right\|^2 f_{V\mathbf{U}}^2\left(v, \mathbf{u}\right) dv d\mathbf{u} \left\{1 + o\left(1\right)\right\},
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{3} \mathcal{K}_{h;12}^{3} W_{h;32} \mathcal{K}_{h;32}\right] \\
& = \Big\langle \int \pi_5\left(x, \mathbf{x}\right) W_h^3\left(y - x\right) \mathcal{K}_h^3\left(\mathbf{y} - \mathbf{x}\right) f_{V\mathbf{U}}\left(x, \mathbf{x}\right) dx d\mathbf{x} \\
& , \pi_5\left(y, \mathbf{y}\right) \int W_h\left(z - y\right) \mathcal{K}_h\left(\mathbf{z} - \mathbf{y}\right) f_{V\mathbf{U}}\left(z, \mathbf{z}\right) dz d\mathbf{z} \Big\rangle f_{V\mathbf{U}}\left(y, \mathbf{y}\right) dy d\mathbf{y} \\
& = \int \Big\langle \pi_5\left(y, \mathbf{y}\right)\left(C_{W\mathcal{K},33} f_{V\mathbf{U}}\left(y, \mathbf{y}\right) h^{-2d} + \psi_{W\mathcal{K},33}\left(h, \left(y, \mathbf{y}\right)\right)\right) \\
& \pi_5\left(y, \mathbf{y}\right)\left[f_{V\mathbf{U}}\left(y, \mathbf{y}\right) + h^P S_{W\mathcal{K},5}\left(y, \mathbf{y}\right) + \beta_{W\mathcal{K},5}\left(h, \left(y, \mathbf{y}\right)\right)\right] \Big\rangle f_{V\mathbf{U}}\left(y, \mathbf{y}\right) dy d\mathbf{y} \\
& = \frac{C_{W\mathcal{K},33}}{h^{-2d}} \int \left\|\pi_5\left(y, \mathbf{y}\right)\right\|^2 f_{V\mathbf{U}}^3\left(y, \mathbf{y}\right) dy d\mathbf{y} \left\{1 + o\left(1\right)\right\}, \text{ and}
\end{aligned}
$$

from Lemma B-3, it also follows

$$
\begin{aligned}
& E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{2} \mathcal{K}_{h;12}^{2} W_{h;31} \mathcal{K}_{h;31} W_{h;32} \mathcal{K}_{h;32}\right] \\
& = E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{2} \mathcal{K}_{h;12}^{2} \times E\left[W_{h;31} \mathcal{K}_{h;31} W_{h;32} \mathcal{K}_{h;32} \middle| \left(V_1, \mathbf{U}_1\right), \left(V_2, \mathbf{U}_2\right)\right]\right] \\
& = \frac{1}{h^d} E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{2} \mathcal{K}_{h;12}^{2} \left\langle W\mathcal{K}, W\mathcal{K}\right\rangle \left(\frac{v_1 - v_2}{h}, \frac{\mathbf{u}_1 - \mathbf{u}_2}{h}\right) f_{V\mathbf{U}}\left(v_1, \mathbf{u}_1\right) \{1 + o(1)\}\right] \\
& = O\left(h^{-2d}\right).
\end{aligned}
$$

Also, from Lemma B-1 it follows

$$
\begin{aligned}
& E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{2} \mathcal{K}_{h;12}^{2} W_{h;41} \mathcal{K}_{h;41} W_{h;32} \mathcal{K}_{h;32}\right] \\
& = E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12}^{2} \mathcal{K}_{h;12}^{2} \times E\left[W_{h;41} \mathcal{K}_{h;41} \middle| \left(V_1, \mathbf{U}_1\right)\right] E\left[W_{h;32} \mathcal{K}_{h;32} \middle| \left(V_2, \mathbf{U}_2\right)\right]\right] \\
& = O\left(h^{-d}\right).
\end{aligned}
$$

By Lemmas B-1 and B-3, all other terms are

$$
\begin{aligned}
& E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12} \mathcal{K}_{h;12} W_{h;31} \mathcal{K}_{h;31} W_{h;32}^{2} \mathcal{K}_{h;32}^{2}\right] \\
& = E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12} \mathcal{K}_{h;12} E\left[W_{h;31} \mathcal{K}_{h;31} W_{h;32}^{2} \mathcal{K}_{h;32}^{2} \middle| \left(V_1, \mathbf{U}_1\right), \left(V_2, \mathbf{U}_2\right)\right]\right] \\
& = \frac{1}{h^{2d}} E\left[\left\langle \pi_{5;1}, \pi_{5;2}\right\rangle W_{h;12} \mathcal{K}_{h;12} \left\langle W\mathcal{K}, W^2\mathcal{K}^2\right\rangle \left(\frac{v_1 - v_2}{h}, \frac{\mathbf{u}_1 - \mathbf{u}_2}{h}\right) f_{V\mathbf{U}}\left(v_1, \mathbf{u}_1\right) \{1 + o(1)\}\right] \\
& = O\left(h^{-2d}\right).
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}\mathcal{K}_{h;12}W_{h;31}\mathcal{K}_{h;31}W_{h;32}\mathcal{K}_{h;32}W_{h;42}\mathcal{K}_{h;42}\right]\\
&=E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}\mathcal{K}_{h;12}\right.\\
&\left.\times E\left[W_{h;31}\mathcal{K}_{h;31}W_{h;32}\mathcal{K}_{h;32}\right|(V_1,\mathbf{U}_1),(V_2,\mathbf{U}_2)\right]E\left[W_{h;42}\mathcal{K}_{h;42}\right|V_2,\mathbf{U}_2\left.\right]\right]\\
&=O\left(h^{-d}\right),
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}\mathcal{K}_{h;12}W_{h;41}\mathcal{K}_{h;41}W_{h;32}^2\mathcal{K}_{h;32}^2\right]\\
&=E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}\mathcal{K}_{h;12}\right.\\
&\left.\times E\left[W_{h;41}\mathcal{K}_{h;41}\right|V_1,\mathbf{U}_1\right]E\left[W_{h;32}^2\mathcal{K}_{h;32}^2\right|V_2,\mathbf{U}_2\left.\right]\right]\\
&=O\left(h^{-d}\right),
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}\mathcal{K}_{h;12}W_{h;31}\mathcal{K}_{h;31}W_{h;42}\mathcal{K}_{h;42}W_{h;52}\mathcal{K}_{h;52}\right]\\
&=E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}\mathcal{K}_{h;12}\right.\\
&\left.\times E\left[W_{h;31}\mathcal{K}_{h;31}\right|V_1,\mathbf{U}_1\right]\left(E\left[W_{h;42}\mathcal{K}_{h;42}\right|V_2,\mathbf{U}_2\right])^3\right]\\
&=O(1)+O\left(h^P\right),
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}^2\mathcal{K}_{h;12}^2W_{h;32}^2\mathcal{K}_{h;32}^2\right]\\
&=E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}^2\mathcal{K}_{h;12}^2E\left[W_{h;32}^2\mathcal{K}_{h;32}^2\right|V_2,\mathbf{U}_2\right]\right]\\
&=O\left(h^{-2d}\right),
\end{aligned}
$$

$$
\begin{aligned}
&E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}^2\mathcal{K}_{h;12}^2W_{h;32}\mathcal{K}_{h;32}W_{h;42}\mathcal{K}_{h;42}\right]\\
&=E\left[\langle\pi_{5;1},\pi_{5;2}\rangle W_{h;12}^2\mathcal{K}_{h;12}^2\left(E\left[W_{h;32}\mathcal{K}_{h;32}\right|V_2,\mathbf{U}_2\right]\right)^2\right]\\
&=O\left(h^{-d}\right),
\end{aligned}
$$

as required. ■

Lemma B-8 *Let Assumptions (A1)–(A3) hold. Then*

$$
\mathcal{B}_{5,VI}=\left\|E\left[\zeta_{51}\right]\right\|^2+O\left(N^{-2}h^{-d}\right)
$$

Proof. Firstly,

$$
\begin{aligned}
&\mathcal{B}_{5,VI} = \\
&\frac{1}{N(N-1)^3} E\left[\left\langle \pi_{5;1}\left\|\sum_{t=3}^{N} W_{h;t1}\mathcal{K}_{h;t1}\right\|^2, \pi_{5;2}\left\|\sum_{s=3}^{N} W_{h;s2}\mathcal{K}_{h;s2}\right\|^2\right\rangle\right] \\
&= \frac{1}{N(N-1)^2} E\left[\langle \pi_{5;1}, \pi_{5;2}\rangle W_{h;31}^2 \mathcal{K}_{h;31}^2 W_{h;32}^2 \mathcal{K}_{h;32}^2\right] \\
&+ \frac{1}{N(N-1)} E\left[\langle \pi_{5;1}, \pi_{5;2}\rangle W_{h;41}\mathcal{K}_{h;41} W_{h;42}\mathcal{K}_{h;42} W_{h;51}\mathcal{K}_{h;51} W_{h;52}\mathcal{K}_{h;52}\right] \\
&+ \frac{1}{N(N-1)} \left\langle E\left[\pi_{5;1} W_{h;31}^2 \mathcal{K}_{h;31}^2\right], E\left[\pi_{5;2} W_{h;32}^2 \mathcal{K}_{h;32}^2\right]\right\rangle \qquad \text{(B-12)} \\
&+ \frac{2}{N} \left\langle E\left[\pi_{5;1} W_{h;31}^2 \mathcal{K}_{h;31}^2\right], E\left[\pi_{5;2} W_{h;42}\mathcal{K}_{h;42} W_{h;52}\mathcal{K}_{h;52}\right]\right\rangle \qquad \text{(B-13)} \\
&+ \left\langle E\left[\pi_{5;1} W_{h;41}\mathcal{K}_{h;41} W_{h;51}\mathcal{K}_{h;51}\right], E\left[\pi_{5;2} W_{h;42}\mathcal{K}_{h;42} W_{h;52}\mathcal{K}_{h;52}\right]\right\rangle. \qquad \text{(B-14)}
\end{aligned}
$$

Now, it follows from Lemma B-3, that

$$
\begin{aligned}
&E\left[\langle \pi_{5;1}, \pi_{5;2}\rangle W_{h;31}^2 \mathcal{K}_{h;31}^2 W_{h;32}^2 \mathcal{K}_{h;32}^2\right] \\
&= E\left[\langle \pi_{5;1}, \pi_{5;2}\rangle E\left[W_{h;31}^2 \mathcal{K}_{h;31}^2 \middle| V_1, \mathbf{U}_1\right] E\left[W_{h;32}^2 \mathcal{K}_{h;32}^2 \middle| V_2, \mathbf{U}_2\right]\right] \\
&= O\left(h^{-d}\right), \text{ and} \\
&E\left[\langle \pi_{5;1}, \pi_{5;2}\rangle W_{h;41}\mathcal{K}_{h;41} W_{h;42}\mathcal{K}_{h;42} W_{h;51}\mathcal{K}_{h;51} W_{h;52}\mathcal{K}_{h;52}\right] \\
&= E\left[\langle \pi_{5;1}, \pi_{5;2}\rangle E\left[W_{h;41}\mathcal{K}_{h;41} W_{h;42}\mathcal{K}_{h;42} \middle| (V_1, \mathbf{U}_1), (V_2, \mathbf{U}_2)\right]\right. \\
&\left.\times E\left[W_{h;51}\mathcal{K}_{h;51} W_{h;52}\mathcal{K}_{h;52} \middle| (V_1, \mathbf{U}_1), (V_2, \mathbf{U}_2)\right]\right] \\
&= O\left(h^{-d}\right).
\end{aligned}
$$

The result follows after noticing that $\|E[\zeta_{51}]\|^2 =$ (B-12) + (B-13) + (B-14). ■

Lemma B-9 *Let Assumptions (A1)–(A3) hold. Then as $h \to 0$,*

$$
\begin{aligned}
\mathcal{B}_{13,I} &= \int \langle \widetilde{\pi}_1(v, \mathbf{u}), \pi_3(v, \mathbf{u})\rangle f_{V\mathbf{U}}^2(v, \mathbf{u})\, dv d\mathbf{u} \{1 + o(1)\}, \\
\mathcal{B}_{13,II} &= \langle \delta_1, \delta_3\rangle + h^P \int \langle \delta_1, \pi_3(v, \mathbf{u})\rangle S_{W\mathcal{K}}(v, \mathbf{u})\, dv d\mathbf{u} + \mathbf{o}\left(h^P\right), \\
\mathcal{B}_{12,I} &= \int \langle \pi_1(\mathbf{u}), \pi_2(\mathbf{u})\rangle f_{\mathbf{U}}^2(\mathbf{u})\, d\mathbf{u} \{1 + o(1)\}, \\
\mathcal{B}_{12,II} &= \langle \delta_1, \delta_2\rangle + h^P \int \langle \delta_1, \pi_2(\mathbf{u})\rangle S_{\mathcal{K}}(\mathbf{u})\, d\mathbf{u} + \mathbf{o}\left(h^P\right).
\end{aligned}
$$

Proof. As before, we show the results for $\mathcal{B}_{12,l}$, for $l = I, II$. From Assumption (A2) and (A3), we have

$$\begin{aligned}
\mathcal{B}_{12,I} &= E\left[\langle \pi_1(\mathbf{U}_1), \pi_2(\mathbf{U}_2)\rangle \mathcal{K}_h(\mathbf{U}_1 - \mathbf{U}_2)\right] \\
&= \int \langle \pi_1(\mathbf{x}), \pi_2(\mathbf{y})\rangle \mathcal{K}_h(\mathbf{x}-\mathbf{y}) f_{\mathbf{U}}(\mathbf{x}) f_{\mathbf{U}}(\mathbf{y})\, d\mathbf{x}d\mathbf{y} \\
&= \int \langle \pi_1(\mathbf{y}+\mathbf{c}h), \pi_2(\mathbf{y})\rangle f_{\mathbf{U}}(\mathbf{y}+\mathbf{c}h) f_{\mathbf{U}}(\mathbf{y}) \mathcal{K}(\mathbf{c})\, d\mathbf{y}d\mathbf{c} \\
&= \int \langle \pi_1(\mathbf{y}), \pi_2(\mathbf{y})\rangle f^2_{\mathbf{U}}(\mathbf{y})\, d\mathbf{y}\left\{1 + o(1)\right\},
\end{aligned}$$

where the last equality comes from the change of variables: $\mathbf{c} = h^{-1}(\mathbf{x}-\mathbf{y})$. Finally, from Lemma B-1, we obtain

$$\begin{aligned}
\mathcal{B}_{12,II} &= E[\langle \pi_1(\mathbf{U}_1), \pi_2(\mathbf{U}_2)\rangle \mathcal{K}_h(\mathbf{U}_3 - \mathbf{U}_2)] \\
&= \int \langle \pi_1(\mathbf{x}), \pi_2(\mathbf{y})\rangle \mathcal{K}_h(\mathbf{z}-\mathbf{y}) f_{\mathbf{U}}(\mathbf{x}) f_{\mathbf{U}}(\mathbf{y}) f_{\mathbf{U}}(\mathbf{z})\, d\mathbf{x}d\mathbf{y}d\mathbf{z} \\
&= \int \left\langle \int \pi_1(\mathbf{x}) f_{\mathbf{U}}(\mathbf{x})\, d\mathbf{x}, \int \pi_2(\mathbf{y}) \mathcal{K}_h(\mathbf{z}-\mathbf{y}) f_{\mathbf{U}}(\mathbf{z})\, d\mathbf{z}\right\rangle f_{\mathbf{U}}(\mathbf{y})\, d\mathbf{y} \\
&= \int \langle \delta_1, \pi_2(\mathbf{y})\rangle f^2_{\mathbf{U}}(\mathbf{y})\, d\mathbf{y} + h^P \int \langle \delta_1, \pi_2(\mathbf{y})\rangle S_{\mathcal{K}}(\mathbf{y})\, d\mathbf{y} + o\left(h^P\right),
\end{aligned}$$

the result follows after noticing that $E\left[\pi_2(\mathbf{y}) f_{\mathbf{U}}(\mathbf{y})\right] = \delta_2$. ■

Lemma B-10 *Let Assumptions (A1)–(A3) hold. Then as* $h \to 0$,

$$\begin{aligned}
\mathcal{B}_{15,I} &= \frac{1}{h^d}\left[C_{W\mathcal{K}} \int \langle \delta_1^*, \pi_5(y,\mathbf{y})\rangle f^2_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y}\left\{1 + o(1)\right\}\right], \\
\mathcal{B}_{15,II} &= \left\langle \delta_1^*, \int \pi_5(y,\mathbf{y}) f^3_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y}\right\rangle + O\left(h^P\right), \\
\mathcal{B}_{15,III} &= \int \langle \widetilde{\pi}_1(y,\mathbf{y}), \pi_5(y,\mathbf{y}) f_{V\mathbf{U}}(y,\mathbf{y})\rangle f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y} + O\left(h^P\right), \text{ and} \\
\mathcal{B}_{15,IV} &= \frac{1}{h^d}\left[C_{W\mathcal{K}} \int \langle \widetilde{\pi}_1(x,\mathbf{x}), \pi_5(x,\mathbf{x}) f_{V\mathbf{U}}(x,\mathbf{x})\rangle f_{V\mathbf{U}}(x,\mathbf{x})\left\{1 + o(1)\right\}\right].
\end{aligned}$$

Proof. Firstly, by Lemma B-2, it follows

$$\begin{aligned}
\mathcal{B}_{15,I} = &\Big\langle \int \pi_1(x,\mathbf{x}) f_{V\mathbf{U}}(x,\mathbf{x})\, dxd\mathbf{x}, \\
&\int \pi_5(y,\mathbf{y}) W_h^2(z-y) \mathcal{K}_h^2(\mathbf{z}-\mathbf{y}) f_{V\mathbf{U}}(z,\mathbf{z})\, dzd\mathbf{z} f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y}\Big\rangle \\
= &\left\langle \delta_1^*, \int \pi_5(y,\mathbf{y})\left[C_{W\mathcal{K}} f_{V\mathbf{U}}(y,\mathbf{y}) h^{-d} + \psi_{W\mathcal{K}}(h,(y,\mathbf{y}))\right] f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y}\right\rangle \\
= &\frac{1}{h^d}\left[C_{W\mathcal{K}} \int \langle \delta_1^*, \pi_5(y,\mathbf{y})\rangle f^2_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y}\left\{1 + o(1)\right\}\right].
\end{aligned}$$

Also, from Lemma B-1, it follows

$$\begin{aligned}
\mathcal{B}_{15,II} &= \left\langle \delta_1^*, \int \pi_5(y,\mathbf{y}) \left[f_{V\mathbf{U}}(y,\mathbf{y}) + h^P S_{W\mathcal{K}}(y,\mathbf{y}) + \beta_{W\mathcal{K}}(h,(y,\mathbf{y}))\right]^2 f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y} \right\rangle \\
&= \left\langle \delta_1^*, \int \pi_5(y,\mathbf{y}) f_{V\mathbf{U}}^3(y,\mathbf{y})\, dyd\mathbf{y} \right\rangle \\
&+ 2h^P \left\langle \delta_1^*, \pi_5(y,\mathbf{y}) S_{W\mathcal{K}}(y,\mathbf{y}) f_{V\mathbf{U}}^2(y,\mathbf{y})\, dyd\mathbf{y} \right\rangle + o\left(h^P\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{B}_{15,III} &= \int \langle \widetilde{\pi}_1(x,\mathbf{x}) W_h(x-y) \mathcal{K}_h(\mathbf{x}-\mathbf{y}) f_{V\mathbf{U}}(x,\mathbf{x})\, dxd\mathbf{x}, \\
&\quad \pi_5(y,\mathbf{y}) W_h(z-y) \mathcal{K}_h(\mathbf{z}-\mathbf{y}) f_{V\mathbf{U}}(z,\mathbf{z})\, dzd\mathbf{z} \rangle f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y} \\
&= \int \langle \widetilde{\pi}_1(y+ch,\mathbf{y}+\mathbf{c}h) W(c) \mathcal{K}(\mathbf{c}) f_{V\mathbf{U}}(y+ch,\mathbf{y}+\mathbf{c}h)\, dcd\mathbf{c}, \\
&\quad \pi_5(y,\mathbf{y}) \left[f_{V\mathbf{U}}(y,\mathbf{y}) + h^P S_{W\mathcal{K}}(y,\mathbf{y}) + \beta_{W\mathcal{K}}(h,(y,\mathbf{y}))\right] \rangle f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y} \\
&= \int \langle \widetilde{\pi}_1(y,\mathbf{y}), \pi_5(y,\mathbf{y}) f_{V\mathbf{U}}(y,\mathbf{y}) \rangle f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y} \\
&\quad + h^P \langle \widetilde{\pi}_1(y,\mathbf{y}), \pi_5(y,\mathbf{y}) S_{W\mathcal{K}}(y,\mathbf{y}) \rangle f_{V\mathbf{U}}(y,\mathbf{y})\, dyd\mathbf{y} + o\left(h^P\right).
\end{aligned}$$

Finally, from Assumption (A3), it follows that $f_{V\mathbf{U}}$ and π_5 are Lipschitz continuous on $\Omega_{V\mathbf{U}}$. Then

$$\begin{aligned}
\mathcal{B}_{15,IV} &= E\left[\langle \widetilde{\pi}_1(V_1,\mathbf{U}_1), \pi_5(V_2,\mathbf{U}_2) W_h^2(V_1-V_2) \mathcal{K}_h^2(\mathbf{U}_1-\mathbf{U}_2) \rangle \right] \\
&= \int \langle \widetilde{\pi}_1(x,\mathbf{x}), \pi_5(y,\mathbf{y}) \rangle \\
&\times W_h^2(x-y) \mathcal{K}_h^2(\mathbf{x}-\mathbf{y}) f_{V\mathbf{U}}(x,\mathbf{x}) f_{V\mathbf{U}}(y,\mathbf{y})\, dxd\mathbf{x}dyd\mathbf{y} \\
&= \frac{1}{h^d} \int \left\langle \widetilde{\pi}_1(x,\mathbf{x}), \int \pi_5(x+ch,\mathbf{x}+\mathbf{c}h) W^2(c) \|\mathcal{K}(\mathbf{c})\| f_{V\mathbf{U}}(x+ch,\mathbf{x}+\mathbf{c}h)\, dcd\mathbf{c} \right\rangle \\
&\times f_{V\mathbf{U}}(x,\mathbf{x})\, dxd\mathbf{x} \\
&= \frac{C_{W\mathcal{K}}}{h^d} \int \langle \widetilde{\pi}_1(x,\mathbf{x}), \pi_5(x,\mathbf{x}) f_{V\mathbf{U}}(x,\mathbf{x}) \rangle f_{V\mathbf{U}}(x,\mathbf{x})\, dxd\mathbf{x} \left\{1+o(1)\right\},
\end{aligned}$$

as required. ■

Lemma B-11 *Let Assumptions (A1)–(A3) hold. Then as $h \to 0$,*

$$\begin{aligned}
\mathcal{B}_{23,I} &= \int \langle \pi_2(\mathbf{y}), \pi_3(y,\mathbf{y}) \rangle f_{\mathbf{U}}(\mathbf{y}) f_{V\mathbf{U}}^2(y,\mathbf{y})\, dyd\mathbf{y} + O\left(h^P\right), \\
\mathcal{B}_{23,III} &= \int \langle \pi_2(\mathbf{y}), \pi_3(y,\mathbf{y}) \rangle f_{\mathbf{U}}(\mathbf{y}) f_{V\mathbf{U}}^2(y,\mathbf{y})\, dyd\mathbf{y} + O\left(h^P\right), \\
\mathcal{B}_{23,IV} &= \frac{1}{h^{d-1}} \left[C_{\mathcal{K}} \int \langle \widetilde{\pi}_2(v,\mathbf{u}), \pi_3(v,\mathbf{u}) \rangle f_{V\mathbf{U}}^2(v,\mathbf{u})\, dvd\mathbf{u} \left\{1+o(1)\right\} \right].
\end{aligned}$$

Proof. By Lemma B-5, it follows

$$\begin{aligned}
\mathcal{B}_{23,I} &= E\left[\left\langle E\left[\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\middle|\,\mathbf{U}_3\right],\right.\right.\\
&\quad \left.\left. E\left[\pi_3\left(V_1,\mathbf{U}_1\right)W_h\left(V_3-V_1\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_1\right)\middle|\,V_3,\mathbf{U}_3\right]\right\rangle\right]\\
&= \int\Big\langle \pi_2\left(\mathbf{y}\right)f_{\mathbf{U}}\left(\mathbf{y}\right)+h^P\widetilde{S}^*_{\mathcal{K}}\left(\mathbf{y}\right)+\widetilde{\beta}^*_{\mathcal{K}}\left(h,\mathbf{y}\right),\\
&\quad \pi_3\left(y,\mathbf{y}\right)f_{V\mathbf{U}}\left(y,\mathbf{y}\right)+h^P\widetilde{S}_{W\mathcal{K}}\left(y,\mathbf{y}\right)+\widetilde{\beta}_{W\mathcal{K}}\left(h,\left(y,\mathbf{y}\right)\right)\Big\rangle f_{V\mathbf{U}}\left(y,\mathbf{y}\right)dyd\mathbf{y}\\
&= \int\left\langle \pi_2\left(\mathbf{y}\right),\pi_3\left(y,\mathbf{y}\right)\right\rangle f_{\mathbf{U}}\left(\mathbf{y}\right)f^2_{V\mathbf{U}}\left(y,\mathbf{y}\right)dyd\mathbf{y}+O\left(h^P\right).
\end{aligned}$$

Also, by Lemmas (B-1) and (B-5), we have

$$\begin{aligned}
\mathcal{B}_{23,III} &= E\left[\left\langle\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right),\pi_3\left(V_2,\mathbf{U}_2\right)\right\rangle\mathcal{K}_h\left(\mathbf{U}_2-\mathbf{U}_1\right)W_h\left(V_3-V_2\right)\mathcal{K}_h\left(\mathbf{U}_3-\mathbf{U}_2\right)\right]\\
&= \int\left\langle\widetilde{\pi}_2\left(x,\mathbf{x}\right),\pi_3\left(y,\mathbf{y}\right)\right\rangle\mathcal{K}_h\left(\mathbf{y}-\mathbf{x}\right)\times\\
&\quad W_h\left(z-y\right)\mathcal{K}_h\left(\mathbf{z}-\mathbf{y}\right)f_{V\mathbf{U}}\left(x,\mathbf{x}\right)f_{V\mathbf{U}}\left(y,\mathbf{y}\right)f_{V\mathbf{U}}\left(z,\mathbf{z}\right)dxd\mathbf{x}dyd\mathbf{y}dzd\mathbf{z}\\
&= \int\Big\langle\int\widetilde{\pi}_2\left(x,\mathbf{x}\right)\mathcal{K}_h\left(\mathbf{y}-\mathbf{x}\right)f_{V\mathbf{U}}\left(x,\mathbf{x}\right)dxd\mathbf{x},\\
&\quad \int\pi_3\left(y,\mathbf{y}\right)W_h\left(z-y\right)\mathcal{K}_h\left(\mathbf{z}-\mathbf{y}\right)f_{V\mathbf{U}}\left(z,\mathbf{z}\right)dzd\mathbf{z}\Big\rangle f_{V\mathbf{U}}\left(y,\mathbf{y}\right)dyd\mathbf{y}\\
&= \int\Big\langle\pi_2\left(\mathbf{y}\right)f_{\mathbf{U}}\left(\mathbf{y}\right)+h^P\widetilde{S}^*_{\mathcal{K}}\left(\mathbf{y}\right)+\widetilde{\beta}^*_{\mathcal{K}}\left(h,\mathbf{y}\right),\\
&\quad \pi_3\left(y,\mathbf{y}\right)\left[f_{V\mathbf{U}}\left(y,\mathbf{y}\right)+h^PS_{W\mathcal{K}}\left(y,\mathbf{y}\right)+\beta_{W\mathcal{K}}\left(h,\left(y,\mathbf{y}\right)\right)\right]\Big\rangle f_{V\mathbf{U}}\left(y,\mathbf{y}\right)dyd\mathbf{y}\\
&= \int\left\langle\pi_2\left(\mathbf{y}\right),\pi_3\left(y,\mathbf{y}\right)\right\rangle f_{\mathbf{U}}\left(\mathbf{y}\right)f^2_{V\mathbf{U}}\left(y,\mathbf{y}\right)dyd\mathbf{y}+O\left(h^P\right).
\end{aligned}$$

Finally, after a change of variables $c=\left(z-x\right)h^{-1}$ and $\mathbf{c}=\left(\mathbf{z}-\mathbf{x}\right)h^{-1}$, we have

$$\begin{aligned}
\mathcal{B}_{23,IV} &= E\left[\left\langle\widetilde{\pi}_2\left(V_1,\mathbf{U}_1\right),\pi_3\left(V_2,\mathbf{U}_2\right)\right\rangle W_h\left(V_1-V_2\right)\left\|\mathcal{K}_h\left(\mathbf{U}_1-\mathbf{U}_2\right)\right\|^2\right]\\
&= \int\left\langle\widetilde{\pi}_2\left(x,\mathbf{x}\right),\pi_3\left(z,\mathbf{z}\right)\right\rangle W_h\left(z-x\right)\left\|\mathcal{K}_h\left(\mathbf{z}-\mathbf{x}\right)\right\|^2f_{V\mathbf{U}}\left(x,\mathbf{x}\right)f_{V\mathbf{U}}\left(z,\mathbf{z}\right)dxd\mathbf{x}dzd\mathbf{z}\\
&= \frac{1}{h^{d-1}}\int\left\langle\widetilde{\pi}_2\left(x,\mathbf{x}\right),\pi_3\left(x+ch,\mathbf{x}+\mathbf{c}h\right)\right\rangle\times\\
&\qquad f_{V\mathbf{U}}\left(x+ch,\mathbf{x}+\mathbf{c}h\right)f_{V\mathbf{U}}\left(x,\mathbf{x}\right)W\left(c\right)\left\|\mathcal{K}\left(\mathbf{c}\right)\right\|^2dxd\mathbf{x}dcd\mathbf{c}\\
&= \frac{1}{h^{d-1}}\int\left\langle\widetilde{\pi}_2\left(x,\mathbf{x}\right),\pi_3\left(x,\mathbf{x}\right)\right\rangle f^2_{V\mathbf{U}}\left(x,\mathbf{x}\right)dxd\mathbf{x}\int\left\|\mathcal{K}\left(\mathbf{c}\right)\right\|^2d\mathbf{c}\left\{1+O\left(h^P\right)\right\},
\end{aligned}$$

as required. ■

Lemma B-12 *Let Assumptions (A1)–(A3) hold. Then*

$$\begin{aligned}
\mathcal{B}_{25,I} &= O\left(h^{-2(d-1)}\right), \\
\mathcal{B}_{25,II} &= O\left(h^{-(d-1)}\right), \\
\mathcal{B}_{25,III} &= O\left(Nh^{-2(d-1)}\right) + O\left(N^2\right) + O\left(N^2 h^P\right), \\
\mathcal{B}_{25,IV} &= O\left(Nh^{-(d-1)}\right) + O\left(N^2\right) + O\left(N^2 h^P\right), \\
\mathcal{B}_{25,V} &= O\left(Nh^{-(d-1)}\right) + O\left(N^2\right) + O\left(N^2 h^P\right), \\
\mathcal{B}_{25,VI} &= O\left(Nh^{-(d-1)}\right) + O\left(N^2\right) + O\left(N^2 h^P\right).
\end{aligned}$$

Proof. Firstly,

$$\begin{aligned}
\mathcal{B}_{25,I} &= E\left[\left\langle \widetilde{\pi}_2\left(V_1, \mathbf{U}_1\right), \pi_5\left(V_2, \mathbf{U}_2\right)\right\rangle W_h\left(V_1 - V_2\right) \mathcal{K}_h^3\left(\mathbf{U}_2 - \mathbf{U}_1\right)\right] \\
&= \int \left\langle \widetilde{\pi}_2\left(x, \mathbf{x}\right), \pi_5\left(y, \mathbf{y}\right)\right\rangle W_h\left(y - x\right) \mathcal{K}_h^3\left(\mathbf{y} - \mathbf{x}\right) f_{V\mathbf{U}}\left(x, \mathbf{x}\right) f_{V\mathbf{U}}\left(y, \mathbf{y}\right) dx d\mathbf{x} dy d\mathbf{y} \\
&= \frac{1}{h^{2(d-1)}} \int \left\langle \widetilde{\pi}_2\left(x, \mathbf{x}\right), \pi_5\left(x + ch, \mathbf{x} + \mathbf{c}h\right)\right\rangle \\
&\qquad \times W\left(c\right) K^3\left(\mathbf{c}\right) f_{V\mathbf{U}}\left(x, \mathbf{x}\right) f_{V\mathbf{U}}\left(x + ch, \mathbf{x} + \mathbf{c}h\right) dc d\mathbf{c} dx d\mathbf{x} \\
&= O\left(h^{-2(d-1)}\right), \text{ by a further Taylor–series expansion.}
\end{aligned}$$

It follows from Lemmas B-1, B-2, B-3, and B-4 that all other terms have the following orders of magnitude:

$$\begin{aligned}
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12}^{2} W_{h;32}\mathcal{K}_{h;32}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12}^{2} E\left[W_{h;32}\mathcal{K}_{h;32}\right| V_2, \mathbf{U}_2]\right] \\
&= O\left(h^{-(d-1)}\right), \\
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}^{2}\mathcal{K}_{h;32}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}^{2} E\left[\mathcal{K}_{h;32}\right| V_2, \mathbf{U}_2]\right] \\
&= O\left(h^{-(d-1)}\right), \\
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12} W_{h;32}\mathcal{K}_{h;32}^{3}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12} E\left[W_{h;32}\mathcal{K}_{h;32}^{3}\right| V_2, \mathbf{U}_2]\right] \\
&= O\left(h^{-2(d-1)}\right), \\
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12}\mathcal{K}_{h;32} W_{h;42}\mathcal{K}_{h;42}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12} E\left[\mathcal{K}_{h;32}\right| V_2, \mathbf{U}_2] E\left[W_{h;42}\mathcal{K}_{h;42}\right| V_2, \mathbf{U}_2]\right] \\
&= O(1) + O\left(h^{P}\right), \\
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12}\mathcal{K}_{h;31} W_{h;32}\mathcal{K}_{h;32}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12} E\left[\mathcal{K}_{h;31} W_{h;32}\mathcal{K}_{h;32}\right| (V_1, \mathbf{U}_1), (V_2, \mathbf{U}_2)]\right] \\
&= O\left(h^{-(d-1)}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12}\mathcal{K}_{h;31} W_{h;42}\mathcal{K}_{h;42}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle \mathcal{K}_{h;12} E\left[\mathcal{K}_{h;31}\right| V_1, \mathbf{U}_1] E\left[W_{h;42}\mathcal{K}_{h;42}\right| V_2, \mathbf{U}_2]\right] \\
&= O(1) + O\left(h^{P}\right),
\end{aligned}$$

$$\begin{aligned}
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}\mathcal{K}_{h;31}\mathcal{K}_{h;32}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12} E\left[\mathcal{K}_{h;31}\mathcal{K}_{h;32}\right| (V_1, \mathbf{U}_1), (V_2, \mathbf{U}_2)]\right] \\
&= O\left(h^{-(d-1)}\right), \text{ and}
\end{aligned}$$

$$\begin{aligned}
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12}\mathcal{K}_{h;31}\mathcal{K}_{h;32}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2}\right\rangle W_{h;12}\mathcal{K}_{h;12} E\left[\mathcal{K}_{h;31}\right| V_1, \mathbf{U}_1] E\left[\mathcal{K}_{h;42}\right| V_2, \mathbf{U}_2]\right] \\
&= O(1) + O\left(h^{P}\right),
\end{aligned}$$

as needed. ■

Lemma B-13 *Let Assumptions (A1)–(A3) hold. Then*

$$\mathcal{B}_{25,VII} = \left\langle E[\widetilde{\zeta}_{21}], E\left[\zeta_{51}\right]\right\rangle + O\left(N^{-2}h^{-(d-1)}\right) + o\left(N^{-1}\right).$$

Proof. Firstly,

$$\begin{aligned}
&\mathcal{B}_{25,VII} \\
&= \frac{1}{N\left(N-1\right)^{2}} E\left[\left\langle \widetilde{\pi}_{2;1} \textstyle\sum_{t=3}^{N} \mathcal{K}_{h;t1}, \pi_{5;2} \left\| \textstyle\sum_{t=3}^{N} W_{h;t2} \mathcal{K}_{h;t2} \right\|^{2} \right\rangle\right] \\
&= \frac{1}{N^{2}} E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2} \right\rangle \mathcal{K}_{h;31} W_{h;32}^{2} \mathcal{K}_{h;32}^{2}\right] \\
&+ \frac{2}{N} E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2} \right\rangle \mathcal{K}_{h;31} W_{h;32} \mathcal{K}_{h;32} W_{h;42} \mathcal{K}_{h;42}\right] \\
&+ \frac{1}{N} \left\langle E\left[\widetilde{\pi}_{2;1} \mathcal{K}_{h;31}\right], E\left[\pi_{5;2} W_{h;42}^{2} \mathcal{K}_{h;42}^{2}\right] \right\rangle && \text{(B-15)} \\
&+ \left\langle E\left[\widetilde{\pi}_{2;1} \mathcal{K}_{h;31}\right], E\left[\pi_{5;2} W_{h;42} \mathcal{K}_{h;42} W_{h;52} \mathcal{K}_{h;52}\right] \right\rangle. && \text{(B-16)}
\end{aligned}$$

Now, it follows from Lemma B-3, that

$$\begin{aligned}
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2} \right\rangle \mathcal{K}_{h;31} W_{h;32}^{2} \mathcal{K}_{h;32}^{2}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2} \right\rangle E\left[\mathcal{K}_{h;31} W_{h;32}^{2} \mathcal{K}_{h;32}^{2} \middle| \left(V_{1}, \mathbf{U}_{1}\right), \left(V_{2}, \mathbf{U}_{2}\right)\right]\right] \\
&= O\left(h^{-(d-1)}\right), \text{ and} \\
&E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2} \right\rangle \mathcal{K}_{h;31} W_{h;32} \mathcal{K}_{h;32} W_{h;42} \mathcal{K}_{h;42}\right] \\
&= E\left[\left\langle \widetilde{\pi}_{2;1}, \pi_{5;2} \right\rangle E\left[\mathcal{K}_{h;31} W_{h;32} \mathcal{K}_{h;32} \middle| \left(V_{1}, \mathbf{U}_{1}\right), \left(V_{2}, \mathbf{U}_{2}\right)\right]\right. \\
&\left. \times E\left[W_{h;42} \mathcal{K}_{h;42} \middle| \left(V_{2}, \mathbf{U}_{2}\right)\right]\right] \\
&= O\left(h\right).
\end{aligned}$$

The result follows after noticing that $\langle\ E[\widetilde{\zeta}_{21}], E\left[\zeta_{51}\right]\ \rangle$ = (B-15) + (B-16). ■